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FORMULATION OF THE NONLINEAR ANALYSIS OF SHELL-LIKE STRUCTURES, SUBJECTED TO TIME-DEPENDENT MECHANICAL AND THERMAL LOADING

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ABSTRACT

The object of the research reported herein was to develop a general mathematical model and solution methodologies for analyzing the structural response of thin, metallic shell structures under large transient, cyclic, or static thermomechanical loads. Among the system responses associated with these loads and conditions are thermal buckling, creep buckling, and ratcheting. Thus geometric and material nonlinearities (of high order) can be anticipated and must be considered in developing the mathematical model. The methodology is demonstrated through different problems of extension, shear and of planar curved beam. Moreover, importance of the inclusion of large strains is clearly demonstrated, through the chosen applications.

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LIST OF SYMBOLS

A	area
dA	area element
$a_{\alpha\beta}$	first fundamental form of the surface
A_{ij}; A^{ij}; $A^{i}_{.j}$	general tensor of rank two in the fix coordinate system
$a_{\alpha\beta}$; $a^{\alpha\beta}$; $a^{\alpha}_{.\beta}$	general tensor of rank two in the material coordinate system
$\bar{B}^{\Gamma}$	body forces per unit area mass
$b_{\alpha\beta}$	second fundamental form of the surface
$\frac{D}{Dt}$, $\dot{\ldots}$	material derivative
$\frac{d}{dt}$	total derivative
$\frac{d^J}{dt}$ $\overset{\nabla}{\ldots}$	Jauman derivative
$\frac{\partial}{\partial t}$	partial derivative
$d_{\alpha\beta}$	deformation rate tensor
E	Young's modulus
E_{AB}	strain tensor
F	yield function
$\bar{F}^{\Gamma}$	external forces per unit area mass
f^{i}	body forces per unit mass
f^{α}_{β}	transformation tensor
G	shear modulus
G^{A}, g_{r}	base vectors

G_{ij}, $g_{\alpha\beta}$	metric tensors
H	mean curvature of the surface
h	half thickness of the shell
h	material constant
J	absolute determinant of the transformation tensor
K	Gaussian curvature of the surface
k, α^{γ}_{δ} , $A^{\alpha\beta}_{\gamma\delta}$	material internal variables
L^{α}_{i}	transformation tensor
ℓ	length (axial)
$M^{\pi\phi}$	stress couples resultant
m	mass
dm	elementary mass
$N^{\pi\phi}$	stress resultant
n	normal to the surface
dP^{i}	force vector
q	specific applied heat
$R^{P}_{.ijk}$ $\tau^{\alpha}_{.\beta\gamma\delta}$	Riemann-Christoffel tensor
S^{ij}	Second Piola Kichhoff stress tensor
s	specific entropy
ds	differential of arc length
T	temperature
T^{i}	surface traction
t^{α}_{β}	deviator of Kirchhoff stress tensor
u	specific internal energy

u_i^α	transformation tensor
u^α	material (convected) coordinate system
V	volume
dV	elementary volume
v^i, v^α	velocity vector
W	mechanical work
W_{rs}	spin tensor
x^i	inertial coordinate system
x_α^i	transformation tensor
y^i	material shell coordinate system
α	thermic expansion constant
γ	area mass density
Γ_{jk}^j $\gamma_{\beta\delta}^\alpha$	Christoffel symbols
$\gamma_{\pi\phi}$	shell kinematics variables
ϵ_{ijk}	permutation tensor
δ	variation symbol
δ_{ij}	Kronecker deltas
θ^Γ	shell coordinate system
λ_β^α	shell parameters
λ	material parameter
μ_Λ^Γ	shifter tensor
ν	elastic constant
ν^i	unit vector normal to the surface
ξ_o	normal to the surface
ρ	mass density

ρ_{α}	surface parameter
$\kappa_{\pi\phi}$	shell kinematic variables
σ^{ij}	Cauchy stress tensor
τ^{α}_{β}	Kirchhoff stress tensor
$\bar{\tau}^{\alpha}_{\beta}$	athermal stress
$\overset{*}{\tau}{}^{\alpha}_{\beta}$	viscous overstress
ϕ	specific free energy (Helmholtz function)
ψ^{α}_{β}	rate of transformation tensor
$\omega_{\alpha\beta}$	spin tensor

1. INTRODUCTION

1.1 Background

The prediction of inelastic behavior of metallic materials at elevated temperatures has increased in importance in recent years. The operating conditions within the hot section of a rocket motor or a modern gas turbine engine present an extremely harsh thermo-mechanial environment. Large thermal transients are induced each time the engine is started or shut down. Additional thermal transients from an elevated ambient occur, whenever the engine power level is adjusted to meet flight requirements. The structural elements employed to construct such hot sections, as well as any engine component located therein, must be capable of withstanding such extreme conditions. Failure of a component would, due to the critical nature of the hot section, lead to an immediate and catastrophic loss in power and thus cannot be tolerated. Consequently, assuring satisfactory long term performance for such components is a major concern for the designer.

Traditionally, this requirement for long term durability has been a more significant concern for gas turbine engines rather than rocket motors. However, with the advent of reusable space vehicles, such as the Space Shuttle, the requirement to accurately predict future performance following repeated elevated temperature operations must now be extended to includ the more extreme rocket motor application. These operating blades to severe thermal transients that result in large inelastic strains, and several types of behavior must be considered. The elevated temperatures can lead to thermal buckling and, in addition, creep can be expected to occur. Thus, a combination of thermal-creep buckling behavior leading to large deflections can be anticipated. Because of the cyclic character of the mechanical and thermal loads, progressive growth or ratchetting effects must also be considered. Thus, geometric and material nonlinearities (of high orders) can be anticipated and must be considered in the mathematical model.

Consequently, the industry is concerned with the behavior of thin shell like structural elements subjected to severe time dependent thermo-mechanical loading. Such thin elements, including beams, rings, arches, plates and shells, are presenting generic types of components, which might be located within or adjacent to the hot section of a rocket or a gas turbine engine.

The experience in the gas turbine engine industry indicates, however, that existing analytic tools are not sufficiently reliable to accomplish this task. State of the art methods for predicting hot section component behavior are generally not sufficiently accurate to perform extended use evaluations.

Under this kind of severe loading conditons, the structural behavior is highly nonlinear due to the combined action of geometrical and physical nonlinearities. On one side, finite deformation in a stressed structure introduces nonlinear geometric effects. On the other side, physical nonlinearities arise even in small strain regimes, whereby inelastic phenomena play a particularly important role. From a theoretical standpoint, nonlinear constitutive equations should be applied only in connection with nonlinear transformation measures (implying both deformation and rotations). However, in almost all of the works in this area, the two identified sources of nonlinearities are always separated. This separation yields, at one end of the spectrum, problem of large response, while at the other end, problems of viscous and/or non-isothermal behavior in the presence of small strain.

Because of the nature of the causes, special care is needed in the selection or development of a constitutive law that includes tme and temperature effects. Although there exists a sizeable body of literature on phenomenological consitutive equations for the rate- and temperature-dependent plastic-deformation of metallic materials, to date rational and thermodynamically consistent elastic-thermoviscoplastic constitutive relations capable of incorporating the effects of large strains and rotations have not been demonstrated.

Constitutive models for small strain in engineering literature may generally be grouped into three categories: classical plasticity, nonlinear viscoelasticity, and theories based on microstructural phenomena. Each group can be further separated into "unified" and "uncoupled" theories, where the two differ in their approach to the treatment of rate - independent and rate-dependent inelastic deformation. The uncoupled theories decompose the inelastic strain rate into a time-independent plastic strain rate and a timedependent creep rate with independent constitutive relations describing plastic and creep behavior. Such uncoupling of the strain components provides for simpler theories to be developed but precludes any creep-plasticity interaction. Recognizing that cyclic plasticity,creep and recovery are not independent phenomena but rather are very interdependent, a number of "unified" models for inherently time-dependent nonelastic deformation have been developed recently.

Classical incremental plasticity theories are macrophenomenological because they base the derivation of state variables purely on experimental results without direct reference to the microstructure of the material. Most incremental plasticity theories have four major components: (1) a stress-elastic strain relations, (2) a yield function describing the onset of plastic deformation, (3) a hardening rule which prescribes the strain-hardening of the material and the modification of the yield surface during plastic flow, and (4) a flow rule which defines the components of strain that are plastic or nonrecoverable. Research in this area is voluminous. For example, Zienkiewicz and Cormeau[1] developed a rate dependent unified theory which allows for nonassociative plasticity and strain softening, but does not model the Bauschinger effect or temperature dependence. Extensions of classical plasticity to model both rate and temperature effects were presented recently by Allen and Haisler[2], Haisler and Cronenworth[3], and Yamada and Sakurai[4].

In the nonlinear viscoelastic approach, the constitutive relation is expressed as a single integral or convoluted form. This type of constitutive model employs the thermodynamic laws along with physical constraints to complete the formulation. A detailed review of several existing theories is presented by Walker[5]. Walker's[5] theory is based on a unified viscoplastic integral developed by modifying the constitutive relations for a linear three paramter viscoelastic solid. The theory contains clearly defined material parameters, a rate dependent equilibrium stress, and a proposed multiaxial model. An important shortcoming of Walker's theory is its failure to model transient temperature conditions. Many other nonlinear viscoelastic theories have been proposed including those by Cernocky and Krempl[6], Valanis[7] and Chabache[8].

The microphenomenological theories attempt to represent the response of polycrystalline materials in terms of various micromechanisms of deformation and failure. Various dislocation theories have been developed to predict plastic deformation in terms of dislocation interaction, slip, glide, density, etc. Most of the material models developed, to date, depend primarily on the number of state variables used and their growth or evolutionary laws. Many of the recent "unified" microphenomenological theories have been discussed and evaluated by Walker[9], and Chan et. al[10].

One example of a microphysically based constitutive law is an elastic-viscoplastic theory based on two internal state variables as proposed by Bodner, et al.[11]. These authors,demonstrate the ability of the constitutive equations to represent the principal features of cyclic loading behavior including softening upon stress reversal, cyclic hardening or softening, cyclic saturation, cyclic relaxation, and cyclic creep. One limitation of the formulation though is that the computed stress-strain curves are independent of the strain amplitude and therefore too "flat" or "square".

Miller[12] has reported research on the modelling of cyclic plasticity with "unified" constitutive equations. He also recognizes the shortcomings of many theories in predicting hysteresis loops, which are oversqaure in comparison to observed experimental behavior. Improvement is accomplished by making the kinematic work-hardening coefficient depend on the back stress and the sign of the nonelastic strain term. Theories that are similar in format to Miller's have been proposed by Krieg, Swearengen and Rhode[13] and by Hart[14]. The models use two internal state variables to reflect current microstructure state and are based upon models for dislocation processes in pure metals. All these constitutive theories were formulated without the use of a yield criterion. Since these models do not contain a completely elastic regime, the function that describes the inelastic strain rate should be such that the inelastic strain rate is very small for low stress levels. Theories with a yield function and a full elastic regime have been developed for the case of isotropic hardening by

Robinson[15], and by Lee and Zavrel[16] for both isotropic and directional hardening.

As previously noted, the quantities utilized in the small strain theory of viscoplasticity (stress, strain, stress rate, and strain rate) are defined only within the assumption of "small strains". Yet the precise definition of what constitutes "small strain" is always left unstated. Whether or not the stresses for a given case are "small" cannot be determined a priori by geometric considerations. In general, one cannot know in advance whether for a given loading of a material the "small strain" assumption (always left undefined) will hold or not. The question of whether the small-strain approximations are valid is always avoided in the "small strain" literature. Furthermore, as Hill[17] points out, the really typical plastic problems involve changes in geometry that cannot be disregarded. In many cases, for example, it is sufficient to take into account finite plastic strains and small elastic strains or vice versa. From the theoretical viewpoint it is desirable in all cases to have a theory which intrinsically allows for both the elastic and plastic strains to be large. Such a theory of course, must reduce to the earlier mentioned special cases, as limiting cases. Furthermore, such theories provide a check for those which are obtained by generalizing small strain theories.

The mathematical theories of deformation and flow of matter deal essentially with the gross properties of a medium. Heat and mechanical work are considered as additional causes for a change of the state of the medium. The resulting phenomena in any particular material are not unrelated. Therefore, a thermodynamical treatment of the foundation of the theory of flow and deformation is appropriate, and indeed the obvious approach. Two very different main approaches to a thermodynamic theory of a continuum can be identified. These differ from each other in the fundamental postulates upon which the theories are based. An essential controversy (a good survey of this controversy is given in Ref. 18) can be traced through the whole discussion of the thermodynamic aspects of continuum mechanics. None of these approaches is concerned with the atomic structure of the material. They, therefore, represent purely phenomenological approximations. Both theories are characterized by the same fundamental requirement that the results should be obtained without having recourse to statical or kinetic methods.

Within each of these approaches there are two distinct methods of describing history and dissipative effects: the functional theory[19], in which all dependent variables are assumed to depend on the entire history of the independent variables, and the internal variable approach[20], wherein history dependence is postulated to appear implicitly in a set of internal variables. For experimental as well as analytical reasons[21,22] the use of internal variables in modeling inelastic solids is gaining widespread usage, in current research. The main differences among the various modern theories lie in the choice of these internal variables.

The predictive value of an elastic- viscoplastic material model for non-isothermal, large deformation analyses depends therefore on three basic elements:

a) the nonlinear kinematic description of the elastic-plastic deformation.

b) thermodynamic considerations

c) the choice of external and internal thermodynamic variables.

as well as on their interactions.

The problem of viscoplastic deformations in shells has been treated at several levels of approximation and generality.

The simplest approaches* are based on the assumption of infinitesimal displacement gradients (which implies infinitesmal strains) and a material model of stationary creep, sometimes with an approximate inclusion of primary creep.

A more general analysis utilizes shell kinematics for moderately large displacement gradients (at least some of them), infinitesimal strains, and material models of stationary or simple non-stationary creep*. Ths type of assumption is capable of solving problems of creep buckling[24], and it does reproduce the sometimes stiffening effect of the interaction between the normal forces and the normal deflection. Extension of these kind of formulation with a viscoplastic material model is presented in Refs. 25, 26, & 27. The use of numerical methods[28] makes possile the solution for many non-trivial types of structures.

The problems of large strains, which arise in the analysis of large creep or thermal deformation of shells, have not been treated at all in a general manner. Recognizing that finite strain effects are present in these problems, reliable prediction demand tht such effects be included rationally and properly in the analysis. In addition to the necessary basic kinemtical and dynamical equations of the shell theory, such an analysis must incorporate a correctly invariant formulation of the material equations and requires an evaluation of the strain-rate tensors through the thickness of the shell. Such an analysis cannot be found in explicit form, at least in the readily accessible engineering literature.

Several authors have developed mathematical description of the kinemation of the three dimensional deformation of elastic or

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* A comprehensive survey of these works is given in Ref. 23.

viscoelastoplastic materials[29,30]. However, it is not clear how to best select to reference space and configuration for the stress tensor, bearing in mind the rheologies of realistic materials. Although an intrinsic relation, which satisfies material objectivity can be used[31,32], the choice is not unique (see for example Refs. 30, 33, 34).

1.2 Purpose of the Present Study

The objectives of the present research are to develop a general mathematical model and solution methodologies for analyzing the structural response of thin, metallic shell-type structures under large transient, cyclic or static thermomechanical loads. Among the system responses, which are associated with these loads and conditions, are thermal buckling, creep buckling and ratchetting. Thus, geometric and material type nonlinearities (of high order) can be anticipated and must be considered in the development of the mathematical model. Futhermore, this must also be accommodated in the solution procedures. The results obtained from this analysis are compared with the available experimental data, as well as with results obtained from "small strain" analyses in order to ascertain the range of validity of the "small strain" approximation.

1.3 Synopsis of the Present Study

Secton 2 contains the concepts that are necessary for the development of a general "finite strain" theory for thin bodies with path-dependent, time-dependent and temperature-dependent material nonlinearities.

A complete true abinito rate theory of kinematics and kinetics for continuum, without any restriction on the magnitude of the strains or the deformations is formulated. The time dependence and large strain behavior are incorporated through the introduction of the time rates of the metric and curvature in two coordinate systems; a fixed (spatial), and a convected (material) one. The details of the reported developments are carried out by using tensor analysis. Special attention is directed to coordinate transformations as applied to continuum mechanics. Consideration of the kinematics of space (i.e., the intrinsic rates of change as they are observed by a geometer within a closed neighborhood of material particles) focused on the distinction between a fixed (spatial) coordinate system and a convected (material) coordinate system. Moreover, the rates of change of tensors are presented in both systems, taking the various tensor and kinematical effects into account. The relations between the time derivative and the covariant derivatives (gradient) are developed, and these illustrate the possibilities of curved space or motion.

The time derivatives are applied to the basic laws of continuum mechanics and thermodynamics to generate the equations of equilibrium rate and those for the compatiblity of the deformation rates. Finally, the principles of the rate of virtual power and the rate of conservation of energy are introduced and employed, and these should provide a basis the development of computational methods.

The general form of the constitutive equations, employed in the analysis, is presented in Section 3.

The metric tensor (time rate of change) in the convected material coordinate system is linearly decomposed into elastic and plastic parts. In this formulation, a yield function is assumed, which is dependent on the rate of change of stress, metric, temperature, and a set of internal variables. Moreover, a hypo-elastic law is chosen to describe the thermo-elastic part of the deformation.

A time and temperature dependent viscoplasticity model is formulated, in this convected material system, to account for finite strains and rotations. The history and temperature dependence are incorporated through the introduction of internal variables. The choice of these variables, as well as their evolution, is motivated by phenomenological thermodynamic considerations. The nonisothermal elastic-viscoplastic deformation process is described completely by "thermodynamic state" equations.

Many pitfalls in the analyses of various investigations are indicated. A very important point that has been consistently neglected by many analysts and computer programs is to indicate precisely in what form the constitutive properties have to be input. Most investigators after an elaborate treatment of a general theory in tensor notation, leave undefined the constitutive equations to be measured in the laboratory. In Subsection 4.1, the homogeneous uniaxial irrotational deformation of a continuum is treated, with at least two purposes in mind: (1) to give a clear physical understanding of the quantities involved in the analysis (which is not possible to obtain through the tensor index notation) and (2) since the most common material test is the uniaxial test, to identify precisely what are the quantities that one should measure in the laboratory (as well as how to express these data to conform with the constitutive equations used in the theoretical material model).

In Sections 5 and 6, the previous developments of Sections 2 and 3 are utilized to derive consistent kinematic, dynamic and constitutive equations, which are valid for finite strains and rotations of thin bodies. Some of these equations seem to be original (have not been found in the literature by the authors).

Five different shell theories (approximations) in rate form, starting with the simple Kirchhoff-Love theory and finishing with a complete unrestricted one, are considered in Section 5. Three different curved and straight beam problems are studied in Section 6. The results from the "finite strain" analysis are compared with the results from the "small strain" theory to ascertain the range of validity of "small strain" theory for the present type of problems. Moreover, the time and temperature dependence and effects of the new constitutive relations are compared with the results of the classical formulations of thermo-elasto-plasticity.

The future research is summerized in Section 7.

2. GENERAL FORMULATION; THREE-DIMENSIONAL CONTINUUM

2.1 Introduction

The inherent difficulties associated with nonlinear continuum mechanics, along with theoretical considerations, lead to the formulation of incremental constitutive equations. Thus, the rate concepts arise and reformulation of the basic laws with the aid of the rate approach becomes essential. As usage of tensor operators is a common practice in the scope of continuum mechanics, their rates need to be formulated rigorously.

It is obvious that the subtleties of tensor analysis and of the rate concepts, distinguishing between the spatial and the material descriptions of the continuum, require careful examination and rigorous formulations; otherwise, inaccuracies are likely to be encountered, especially in conjunction with the derivation of the rate of a gradient.

The objectives of this section are then the systematic formulation of the rates of tensor operators, yielding integral and differential rate theorems. Resultant theorems are definitely meaningful. Moreover, the consistent considerations throughout the process of formulation are vital ingredients for gaining insight, in anticipation of further developments, when dealing with special structural elements.

Treatises that have influenced this write-up are attributed to Truesdell et al.[35-37], Sedov[38-40], Green et al.[41-43], Sokolnikoff[44] and McConnell[45].

The tasks of the reported developments are carried out with the aid of tensor analysis. Tensor definitions, notations, theorems, corollaries, etc. are rephrased in Subsection 2.2. This brief refresher on tensor analysis follows Sokolnikoff[44] and McConnell[45]. Special attention is drawn to transformation of coordinates, bearing in mind the applications of continuum mechanics.

Subsection 2.3 is dedicated to the kinematics of space, i.e., the intrinsic rates of change, as they are observed by a geometer within the closed neighborhood of material particles. The distinction between coordinate systems (essential cornerstone of tensor analysis) is specialized to the one between the fixed (spatial) system and the convected (material) one. Only the Lagrangian description is discussed because the Eulerian one does not seem adequate for solids. The geometric rates are examined in detail, as long as the geometer, whose explorations focus upon the material system, is capable of observing. This subject has been thoroughly elaborated upon by Durban and Baruch[46], whose discipline follows the original ones of Gibbs[47] or Bloch[48], and by Mendelssohn and Baruch[49] whose discipline in turn follows the work of Aris[50]. The latter two illuminate the facets of the various time derivatives (namely, rates of change) with the help of the conventional tensor notation. This approach will be followed, herein.

Subsection 2.4 consists of the development of the rates of tensor components. Numerous publications (including textbooks), especially those oriented toward fluid mechanics, have dealt with the problem of rate of change of tensor components. Since the present work is concerned with solid mechanics, the various derivatives are going to be associated with material points and not with the spatial ones. Unlike the conventions of fluid mechanics (cf. Aris[50]) and following the ideas summarized by Durban and Baruch[46], and by Mendelssohn and Baruch[49], rates of change must be distinguished by the location of the observer.

The intrinsic description is insufficient, as some basic laws refer to rates, which are observed within the fixed system. The partial time derivative and the total derivatives, although defined with respect to the fixed system, are formulated in the material system in a manner that manifests their tensor character. Besides, rigorous formulation of the time derivative of a covariant derivative (gradient) enables the classification of problems with regard to the curvature of space or motion. The time derivatives are applied to the basic laws of continuum mechanics, to obtain the equtions of equilibrium rate and the compatibility of deformation rates. Finally, the principle of the rate of virtual power is derived.

2.2 Geometry and Statics of the Continuum

Let the n-dimensional space be decribed by two systems of coordinates x^i and u^α (Fig. 2.1)

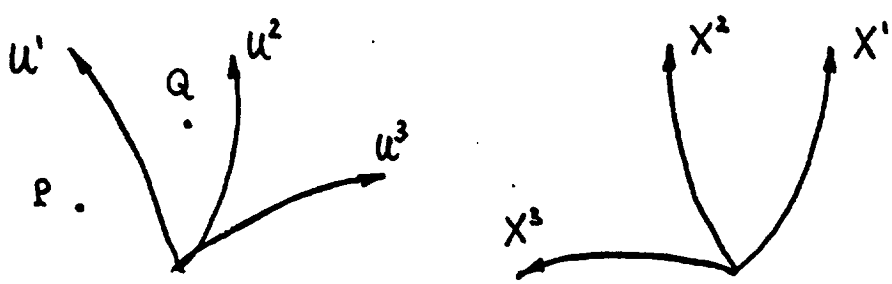

Fig. 2.1: System of Coordinates

Point P can be characterized either by its own coordinates $x^i(P)$ or by $u^\alpha(P)$. The same holds true for the neighboring point Q. As for the arc PQ, it is characterized by the differences Δx^i and Δu^α. When Q approaches

P, differences are replaced by differentials and the following transformation stems

$$dx^i = \frac{\partial x^i}{\partial u^\alpha} du^\alpha \qquad ; \qquad du^\alpha = \frac{\partial u^\alpha}{\partial x^i} dx^i \tag{2.2.1}$$

where dx^i and du^α are the components of the differential of the position vector. In Eq. (2.2.1), and from now on, the summation convention is implemented, i.e., a repeated (dummy) index is summed over its entire permissible range (from 1 to n). Any set of n components, assuming the values a^α when observed in the system $(u^1,\ldots,u^n)$ and the values A^i when observed in the other system, and which obeys a transformation law similar to Eq. (2.2.1),

$$A^i = \frac{\partial x^i}{\partial u^\alpha} a^\alpha \qquad ; \qquad a^\alpha = \frac{\partial u^\alpha}{\partial x^i} A^i, \tag{2.2.2}$$

is defined as "the contravariant components of a vector". Distinction between the systems of coordinates is made by using capital (small) letter and Latin (Greek) indices for quantities as they are observed in the x^i-system and u^α-system, respectively.

If there exists a scalar function ϕ which is point dependent, then, its gradient obeys another transformation law, namely

$$\frac{\partial \phi}{\partial u^\alpha} = \frac{\partial \phi}{\partial x^i} \frac{\partial x^i}{\partial u^\alpha} \tag{2.2.3}$$

Any set of n components a_α snf A_i, obeying the transformation law

$$A_i = \frac{\partial u^\alpha}{\partial x^i} a_\alpha \qquad ; \qquad a_\alpha = \frac{\partial x^i}{\partial u^\alpha} A_i, \tag{2.2.4}$$

is defined as "the set of covariant components of a vector". It should be noted that the components of the position vector (x^i or u^α) often do not constitute components of a vector, according to the definitions of Eqs. (2.2.2) and(2.2.4). It is obvious that "superscript" indices denote contravariant components and "subscript" indices covariant ones. Moreover, it can be shown that summation conventions are applicable in a "diagonal" manner.

The metric properties of the space are determined by the length of the elementary arc (as Q approaches P; see Fig. 2.1) in differential form

$$ds^2 = G_{ij} dx^i dx^j \tag{2.2.5}$$

where ds is the arc length. The quadratic form, Eq. (2.2.5), is specified by the elements (G_{ij}) of a symmetric positive definite matrix. If the space can be described by a system (say y^i) of rectangular Cartesian coordinates, the form of Eq. (2.2.5) reduces to

$$ds^2 = dy^i dy^i = \delta_{ij} dy^i dy^j \tag{2.2.6}$$

where δ_{ij} are the Kronecker deltas. However, points are observed in the u^α-system too, hence

$$\begin{aligned} ds^2 &= g_{\alpha\beta} du^\alpha du^\beta \quad \text{and} \\ g_{\alpha\beta} &= \frac{\partial x^i}{\partial u^\alpha} \frac{dx^j}{du^\beta} G_{ij} \end{aligned} \tag{2.2.7}$$

The sets G_{ij} and $g_{\alpha\beta}$ are defined as the covariant components of the metric tensor. It is symmetric tensor of rank two. Any set of n^2 components, obeying the transformation rules

$$G^{ij} = \frac{\partial x^i}{\partial u^\alpha} \frac{dx^j}{du^\beta} a^{\alpha\beta}$$

$$A^i_{\cdot j} = \frac{\partial x^i}{\partial u^\alpha} \frac{\partial u^\beta}{\partial x^j} \; a^\alpha_{\cdot\beta} \tag{2.28}$$

$$A_{ij} = \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j} \; a_{\alpha\beta}$$

is defined as the set of contravariant (mixed, convariant) components of a tensor of rank two. A tensor is symmetric if it is not affected by an interchange within a pair of indices (i.e., $A_{ij} = A_{ji}$, $A^{ij} = A^{ji}$, etc.). The dots in the mixed components help to distinguish between $a^{\alpha\cdot}_{\cdot\beta}$, $a_{\alpha\cdot}^{\cdot\beta}$,etc.

The conravariant components of the metric tensor are readily available by matrix inversion, namely

$$G^{ij}G_{jr} = \delta^i_{\cdot r}, \qquad g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma \tag{2.2.9}$$

It is obvious that the mixed components of the metric tensor are the Kronecker deltas. The metric tensor facilitates the following operations:

<u>raising indices</u>

$$a^\alpha = g^{\alpha\beta}a_\beta; \; a^{\alpha\beta} = g^{\alpha\rho}g^{\beta\sigma}a_{\rho\sigma}; \; a^\alpha_{\cdot\beta} = g^{\alpha\rho}a_{\rho\beta}, \tag{2.2.10}$$

<u>lowering indices</u>

$$a_\alpha = g_{\alpha\beta}a^\beta; \; a_{\alpha\beta} = g_{\alpha\zeta}g_{\beta\sigma}a^{\zeta\sigma}; \; a_{\alpha\cdot}^{\cdot\beta} = g_{\alpha\rho}a^{\rho\beta} \tag{2.2.11}$$

inner product

$$a^{\alpha}b_{\alpha} = g^{\alpha\beta}a_{\alpha}b_{\beta} = g_{\alpha\beta}a^{\alpha}b^{\beta} = a_{\alpha}b^{\alpha} \qquad (2.2.12)$$

No transformation rule is violated by Eqs. (2.2.10) and (2.2.11).

From now on, the mixed components of a tensor are going to be witten without dots, as no raising or lowering is intended. The idea is to keep track of the omitted dots and remember that their loction is identical in any system of coordinates. Then, the set of n^{p+q} components, obeying the transformation rule

$$A^{i_1 \ldots i_p}_{j_1 \ldots j_q} = \frac{\partial x^{i_1}}{\partial u^{\alpha_1}} \cdots \frac{\partial u^{\beta_1}}{\partial x^{j_1}} \cdots \frac{\partial u^{\beta_q}}{\partial x^{j_q}} a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \qquad (2.2.13)$$

is defined as the set of the covariant (rank p) and covariant (rank q) components of a tensor. Raising or lowering an index affects the nature of the transformation, Eq. (2.2.13).

It often occurs that although a set of indexed quantities can be obtained, the verification of the tensor character of such a set by inspection (whether the set follows the transformation law of Eq. (2.2.13)) may be quite cumbersome. The inconvenience of the verfification of the transformation laws can be circumvented with the aid of the quotient rule, as follows: let $A(i_1 \ldots . i_p, j_1 \ldots . j_q)$ denote a set of n^{p+q} quantitties, the indices of which are i_1 through i_p and j_1 through j_q, and let $B^{j_1 \; \ldots . \; j_s}_{i_1 \; \ldots . \; i_r}$ denote the components (contravariant of rank s and covarariant of rank r, $(r < p, s < q)$ of an arbitrary tensor, if

$$A\left(i_1 \ldots , \; i_p ; \; j_1 \ldots j_q\right) \; B^{j_1 \ldots j_s}_{j \; \ldots i} = C^{i_1 \ldots i_{p-r}}_{j \; \ldots} \qquad (2.2.14)$$

where the C's in Eq. (2.2.14) are the components of a tensor, and the A's are also components of a tensor, namely $A_{j_1}^{i_1} \cdots {}_{j_1}^{i_p}$. This rule is very useful in the course of tensor algebra and tensor calculus as relations such as Eqs. (2.2.14) are often by-products of various derivations. Thus, tensor characteristics can be easily verified (or identified).

It is often required to calculate n-tiple integrals over a specified domain in the n-dimensional space. The n-tiple "volume" element is calculated as follows: let the n-tiple parallelopiped be constructed from a vertex at point P, the edges along the coordinate lines (dx^i for the i-th edge) and the opposite vertex at point Q, the coordinates of which are obtained by adding dx^i to the coodinates of P: then, the volume dV is

$$dV = \sqrt{G}\ dx^1 \ldots dx^n \tag{2.2.15}$$

where G is the determinant of the square matrix, the elements of which are the covariant components of the metric tensor.

Christoffel symbols of the second kind are defined by the formula

$$\Gamma_{jk} = \frac{1}{2} G^{is} \left(\frac{\partial G_{js}}{\partial x^k} + \frac{\partial G_{ks}}{\partial x^j} - \frac{\partial G_{jk}}{\partial x^s} \right) \tag{2.2.16}$$

These symbols are not tensor components. However, Christoffel derived a transformation rule

$$\frac{\partial^2 x^r}{\partial u^\alpha u^\beta} = V^\rho_{\alpha\beta} \frac{\partial x^r}{\partial u^\rho} - \Gamma^r_{jk} \frac{\partial x^j}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\beta} \tag{2.2.17}$$

where the Γ's denote the Christoffel symbols, derived from the g's in the u^α-system according to Eq. (2.2.16). <u>It is obvious that Christoffel symbols do not transform like tensor components unless the coordinate</u> transforma<u>tion (between the u's and the x's) is affine</u>.

There are several tensor operations, in order to produce tensors, e.g,: addition (tensors of same rank), outer products (without repeated indices), contractions (as implied in Eq. 2.2.14)), etc.. All these operations are algebraic in nature. However, tensor fields (i.e., the components depend on the coordinates) can be operated upon bydifferentiation in order to produce tensors of higher ranks. It can be shown that the covariant derivative of a tensor, defined as follows

$$A^{i_1 \ldots i_p}_{i_1 \ldots j_q,k} = \frac{\partial}{\partial x^k} A^{i_1 \ldots i_p}_{j_1 \ldots j_q} + \sum_{\lambda=1}^{p} \Gamma^{i_\lambda}_{r_k} A^{i_1 \ldots r \ldots i_p}_{j_1 \ldots j_q}$$

$$- \sum_{\tau=1}^{q} \Gamma^{r}_{j_\lambda k} A^{i_1 \ldots i_p}_{j_1 \ldots r \ldots j_q} \tag{2.2.18}$$

does actually transform according to Eq. (2.2.13) as a set of components contravariant of rank p and covariant of rank q + 1. The comma in Eq. (2.2.18) denotes the covariant derivatives not to be confused with partial derivatives with respect to the spatial coordinates. The combinmation of dots and of the index r in Eq. (2.2.18) means that the repeated (dummy) index replaces the λth subscript (or superscript), for the sake of contraction. It should be noted that the covariant derivatives of a tensor really show how it varies while the observer is moving along a coordinate line. Hence, the operators of vector analysis (gradient, divergence, curl, etc.) have to be phrased in terms of covariant derivatives.

As a consequence of the definition, given by Eq. (2.2.18), it can be verified that covariant differentiation of sums and products should be accomplished in the same manner as partial differentiation does. Moreover, Ricci's lemma

$$g_{\alpha\beta,\nu} = g^{\alpha\beta}{}_{,\nu} = 0 \tag{2.2.19}$$

suggests that the components of the metric tensor behave like constants as long as covariant differentiation is concerned., Thus, the operations of raising and lowering indices and (on the other hand) of covariant differentation are commutative.

Naturally, covariant derivatives of higher order are obtained systematically, i.e.,

$$A^{i_j \dots i_p}_{j_1 \dots h_{q,kl}} = \left(A^{i_1 \dots i_p}_{j_1 \dots j,_{q,k}}\right)_{,l} \tag{2.2.20}$$

The order of differentation may be significant, as

$$A_{i,jk} - A_{i,kj} = R^{p}_{\cdot ijk} A_p \tag{2.2.21}$$

where the R's are the component of the tensor of Riemann-Christoffel namely

$$R^p_{\cdot ijk} = \frac{\partial}{\partial x^j} \Gamma^p_{ik} - \frac{\partial}{\partial x^k} \Gamma^p_{ij} + \Gamma^s_{ik} \Gamma^p_{sj} - \Gamma^s_{ij} \Gamma^p_{sk} \tag{2.2.22}$$

This tensor manifests the curvatures of space. If and only if, the Riemann-Christoffel tensor components vanish identically, then the space is Euclidean (i.e., can be represented by a system of Cartesian coordinates).

The transformation tensor x^i_α is defined by

$$x^i_\alpha \triangleq \frac{\partial x^i}{\partial u^\alpha} \tag{2.2.23}$$

It can be proven that the quantities x^i_α transform like the contravariant components of a vector in the Latin indices (say, if the x-coordinates transform to another system of x-coordinates) and like the covariant components of a vector in the Greek idices (if the u-coordinates transform to another material system of u-coordinates). Therefore, the symbol x^i_α is written (following Sokolnikoff Ref. [44]) with a Latin superscript and a Greek subscript. A similar reasoning yields the definition of the inverse transformation tensor

$$u_i^\alpha \overset{\Delta}{=} \frac{\partial u^\alpha}{\partial x^i} \tag{2.2.24}$$

A treatment of double tensor, such as x_α^i, requires a definition of total covariant derivative or the tensor derivative. Let a double tensor be given by

$$A^{\alpha\ldots\beta\ i\ldots j}_{\gamma\ldots\delta\ k\ldots l}\,(x^i, u^\alpha) \tag{2.2.25}$$

Its total covariant derivative is defined by

$$A^{\alpha\ldots\beta\ i\ldots j}_{\gamma\ldots\delta\ k\ldots l;\epsilon} = A^{\alpha\ldots\beta\ i\ldots j}_{\gamma\ldots\delta\ k\ldots l,\epsilon} + A^{\alpha\ldots\beta\ i\ldots j}_{\gamma\ldots\delta\ k\ldots l,m} \cdot x^m_\epsilon$$

$$\text{where,}\quad A^{\alpha\ldots\beta\ i\ldots j}_{\gamma\ldots\delta\ k\ldots l,\epsilon} = \frac{\partial A^{\alpha\ldots\beta\ \ i\ldots j}_{\gamma\ldots\delta\ \ k\ldots l}}{\partial u^\epsilon} + \gamma^\alpha_{\rho\epsilon}\, A^{\rho\ldots\beta\ i\ldots j}_{\gamma\ldots\delta\ k\ldots l} + \ldots \tag{2.2.27}$$

$$- \gamma^\rho_{\gamma\epsilon}\, A^{\alpha\ldots\beta\ i\ldots j}_{\rho\ldots\delta\ k\ldots l} + \ldots$$

Thus, it is a partial covariant derivative with respect to u^α, where x^i are constant. For tensors which are not double, the definition given by Eq. (2.2.26) reduces to the regular one.

The transformation tensor x_α^i fulfills (x^i depends on u^α only),

$$\frac{\partial x^i}{\partial u^2} = x^i_{;\alpha} \overset{\Delta}{=} x^i_\alpha \tag{2.2.28}$$

The tensor formulation of the basic equations of continuum mechanics has been presented in numerous books, e.g., Fung[51], Sokolnikoff[44], Green and Zerna[41], etc. Let the three-dimensional Euclidean space (i.e., our empirical engineering space) be described by the system of coordinates x^i.

The solid continuum occupies a definite volume in space, the volume element is denoted by dV. The continuum is bounded by a surface, the area element is denoted by dA, ν^i denote the contravariant components of the unit normal to the surface, i.e., a unit vector that is perpendicular to the surface, namely

$$G_{ij}\nu^i\nu^j = 1 \tag{2.2.29}$$

An area element consists of specific material particles. Let γ stand for their density. This concept is analogous to that of mass density, however, it is intended to specify a measure for the material entities and not the mass of inertial considerations. Since no dynamics are involved, let γ be called the "area mass density" and let dm be the "mass of an area differential element, i.e.,

$$dm = \gamma \; dA \tag{2.2.30}$$

If dP^i denote the components of the force vector, acting on the area element dA, then the surface tractions are defined as the mass density of that force vector

$$T^i = \frac{dp^i}{dm} \; ; \; T^i = \frac{dp^i}{\gamma dA} \; ; \gamma T^i = \frac{dp^i}{dA} \tag{2.2.31}$$

The last equality of Eqs. (2.2.31) defines the "conventional" surface tractions. The present definition, Eqs. (2.2.31), is based solely on grounds of convenience in view of the foreseen kinematic developments.

The contravariant components of the stress tensor, as defined by Cauchy (cf. Fung[51], Sokolnikoff[44], etc.) are denoted by σ^{ij}, and they are associated with the tractions and with the unit normals as follows

$$\gamma T^j = \sigma^{ij}\nu_i \tag{2.2.32}$$

Thus, it is obvious that the stress components are associated with a specific volume, whereas the components of the second Piola-Kirchhoff stress tensor (cf. Hibbit et al.[52])

$$S^{ij} = \frac{\rho}{\rho_o} \sigma^{ij} \qquad (2.2.33)$$

are associated with mass (material entities) as stated in conjunction with the traction, where ρ and ρ_o stand for the current mass density and for a reference one, respectively.

As long as the "classical" terminology of tensor analysis is concerned (cf. Sokolnikoff[44]), the components of the stress tensor (σ^{ij}) are those of absolute tensor, while those of the Piola-Kirchhoff stress tensor (S^{ij}) constitute a relative tensor.

Let f^j denote the components of the body forces per unit mass. The equations of equilibrium are then

$$\sigma^{ij}{}_{,i} + \rho f^j = 0 \qquad (2.2.34)$$

accompanied by the boundary conditions, Eqs. (2.2.32) at the tracted boundaries.

These equations are based on equilibrium considerations exclusively. The kinematic constraints are systematically incorporated by the principle of virtual power (several publications name it "The Principle of Virtual Velocity"). Let δv_j denote the components of the vector of virtual velocity, i.e., a vector field of values of velocity at which the material particles are capable of moving, obeying all kinematical restrictions (continuity, prescribed velocities where the virtual ones vanish, etc.). Then, the integral theorem

$$\int_V \sigma^{ij}\delta V_{j,i}dV - \int_V \rho f^j \delta v_j dV - \int_A \gamma T^j \delta v_j dA = 0 \qquad (2.2.35)$$

is equivalent to the equations of equilibrium and to the complete set of boundary conditions (Note that $\delta v_j = 0$ implies that v_j is prescribed).

2.3. Kinematics of the Continuum

Let a continuum be described by the two coordinate systems of Fig. 2.1 The x^i-system stays at rest and will be labeled as "the fixed system", while the u^α-system is associated with the material points and will be labeled as "the material system". As the continuum is moving and deforming, coordinates x^i of a material point (say P) are changing, i.e., $x^i(P)$ are time-dependent. Yet, a material point preserves its identity and hence its material coordinates (u_α) are not changing in time (t). The coordinate lines are assumed to "deform" with the continuum in order to enable the material points to keep their coordinates (in the u^α-system) unchanged.

The components of the velocity vector in the fixed system are defined by:

$$v^i = \frac{\partial x^i}{dt} \qquad (2.3.1)$$

While a geometer at P is observing, he cannot recognize any change in the coordinates of any point, however, the main purpose of a geometer's interest, namely distances, are obviously changing. As a matter of fact, the differential form of Eq. (2.2.7) should be examined. To do so, following Ref. 35, we must define material derivatives.

Let $A^{i....j}_{k....l}{}^{\alpha....\beta}_{\gamma....\delta} = A^{i....j}_{k....l}{}^{\alpha....\beta}_{\gamma....\delta}\left(x^i, u^\alpha, t\right)$ be some double tensor depending on both x^i and u_α. Then its material derivative $\frac{DA^{\cdots}_{\cdots}}{Dt}$ is given by

$$\frac{DA^{i\ldots j\ \alpha\ldots\beta}_{k\ldots l\ \gamma\ldots\delta}}{Dt} = \left.\frac{\partial A^{i\ldots j\ \alpha\ldots\beta}_{k\ldots l\ \gamma\ldots\delta}}{\partial t}\right|_{\substack{u^\alpha = \text{const}\\ x^i = \text{const}}} + \left\{\frac{\partial A^{\cdots}_{\cdots}}{\partial x} + \Gamma^i_{sq} A^{s\ldots j\ \alpha\ldots\beta}_{k\ldots l\ \gamma\ldots\delta}\right.$$

$$\left. + \ldots - \Gamma^s_{kq} A^{i\ldots j\ \alpha\ldots\beta}_{s\ldots l\ \gamma\ldots\delta} - \ldots\right\} v^q = \left.\frac{\partial A^{\cdots}_{\cdots}}{\partial t}\right|_{\substack{u^\alpha = \text{const}\\ x^i = \text{const}}} + A^{\cdots}_{\cdots,q}\ v^q \tag{2.3.2}$$

The general definition of Eq. (2.3.2) has two particular cases:

A) When one uses only the fixed spatial description for the description of tensor A, i.e., $A = A(x^i,t)$ (as is common in fluid mechanics), then its material derivative, Eq. (2.3.2), becomes

$$\frac{DA^{i\ldots j\ \alpha\ldots\beta}_{k\ldots l\ \gamma\ldots\delta}}{Dt} = \left.\frac{\partial A^{\cdots}_{\cdots}}{\partial t}\right|_{x^i = \text{const}} + A^{\cdots}_{\cdots,q}\ v^q \tag{2.3.3}$$

The second term in the right hand side of Eq. (2.3.3) is called the translation term of $A^{\cdots}_{\cdots}$.

B) When one uses only a material description for A, i.e., $A = A(u^\alpha,t)$ (as is common in continuum mechanics) the material derivative becomes

$$\frac{DA^{i\ldots j\ \alpha\ldots\beta}_{k\ldots l\ \gamma\ldots\delta}}{Dt} = \left.\frac{\partial A^{\cdots}_{\cdots}}{\partial t}\right|_{u^\alpha = \text{const}} + \left\{\Gamma^i_{sq} A^{s\ldots j\ \alpha\ldots\beta}_{k\ldots l\ \gamma\ldots\delta} + \ldots - \Gamma^s_{kq} A^{i\ldots j\ \alpha\ldots\beta}_{s\ldots l\ \gamma\ldots\delta} - \ldots\right\} v^q \tag{2.3.4}$$

In this case, $\frac{D(.)}{Dt}$ is also called an intrinsic derivative of A. It is worthwhile to notice that, for single tensors in the material description, the definition in Eq. (2.3.4) reduces to a simple partial derivative:

$$\frac{DA^{\alpha\ldots\beta}_{\gamma\ldots\delta}}{Dt} = \left.\frac{\partial A^{\alpha\ldots\beta}_{\gamma\ldots\delta}}{\partial t}\right|_{u^\alpha = \text{const.}} \tag{2.3.5}$$

For the velocity vector, v^i, one can show that the definition given by Eq. (2.3.1) can be replaced in the following manner

$$v^i = \frac{dx^i}{dt} = \frac{\partial x^i}{\partial t} = \frac{Dx^i}{Dt} \tag{2.3.6}$$

From here on, all tensors will be given in the material description only, unless otherwise specified. So all material derivatives of the different tensors will be given by Eq., (2.3.4) (for the most general case of double tensors).

Differentiation of Eq. (2.2.7) yields

$$\frac{D}{Dt}\left(ds^2\right) = 2ds\,\frac{D}{Dt}\left(ds\right) = \left(\frac{D}{Dt}\,g_{\alpha\beta}\right) du^\alpha du^\beta \tag{2.3.7}$$

By dividing both sides of Eq. (2.3.7) by $2ds^2$, one obtains

$$\frac{D}{Dt}\left(\log ds\right) = \frac{1}{2}\left(\frac{D}{Dt}\,g_{\alpha\beta}\right)\frac{du^\alpha}{ds}\,\frac{du^\beta}{ds} \tag{2.3.8}$$

Obviously, $\frac{du^\alpha}{ds}$ is a unit vector, and Eq. (2.3.8) expresses the logarithmic deformation rate. The clue for the intrinsic rates of change may be unraveled, then, by the derivation of the rates of change of the metric tensor. Hence, the second of Eqs. (2.2.7) should be differentiated to yield

$$\frac{\partial}{\partial t}\left(g_{\alpha\beta}\right) = \frac{D}{Dt}\left(x^i_\alpha x^j_\beta G_{ij}\right) \tag{2.3.9}$$

As attention is drawn toward the differentiation of Eq. (2.3.9), it should be noticed that the components of the transformation tensor are associated with a material point (they are changing while the continuum is moving). Nevertheless, the components of the metric tensor are associated with a spatial point and their sole cause of variation is the variation of the x-coordinates. Hence, they are inherently independent of time. Then, time differentiation yields

$$\frac{\partial}{\partial t} g_{\alpha\beta} = G_{ij}\left(\frac{Dx^i_\alpha}{Dt} x^j_\beta + x^i_\alpha \frac{Dx^j_\alpha}{Dt} \right) \tag{2.3.10}$$

The time derivatives of x^i_α follows the general form of Eq. (2.3.4)

$$\frac{Dx^i_\alpha}{Dt} = \frac{\partial x^i_\alpha}{\partial t} + \Gamma^i_{sk}\, x^i_{sk}\, x^s_\alpha v^k = \frac{\partial}{\partial t}\left(\frac{\partial x^i}{\partial u^\alpha} \right) + \Gamma^i_{sk} x^s_\alpha v^k$$

$$= \frac{\partial}{\partial u^\alpha}\left(\frac{\partial x^i}{\partial t} \right) + \Gamma^i_{sk}\, x^s_\alpha\, v^k \tag{2.3.11}$$

The last equality in Eq. (2.3.11) is justified by the fact that material coordinates (u^α) are time-independent. Use of Eq. (2.3.6) into Eq. (2.3.11) and of the definition of covariant derivatives, Eq. (2.2.26) yields

$$\frac{Dx^i_\alpha}{Dt} = \frac{\partial}{\partial u^\alpha}\left(v^i\right) + \Gamma^i_{sk}\, v^k x^s_\alpha = v^i{}_{;\alpha} \overset{\Delta}{=} v^i{}_{,\alpha} \tag{2.3.12}$$

Since v^i denote components of a vector in the fixed system, they can be transformed to the material system by Eq. (2.2.2)

$$v^i = x^i_\alpha\, v^\alpha \quad ; \quad v^\rho = u^\rho_r v^r \tag{2.3.13}$$

Naturally, the velocities in the material system do <u>not</u> measure time derivatives of coordinates; their sole significance stems from the transformation rule, Eq. (2.3.13), as the geometer cannot measure them directly (this argument will be dwelt upon later). Hence, thusly operating on Eq. (2.3.12) yields

$$v^i_{,a} = (x^i_\rho v^\rho)_{,\alpha} = x^i_{\rho,\alpha} v^\rho + x^i_\rho v^\rho_{,\alpha} \tag{2.3.14}$$

Special treatment is required for the evaluation of $x^i_{\rho,\alpha}$. As already stated, x^i_α are tensor components in both coordinates, ρ & α systems, then by Eq. (2.2.26)

$$x^i_{\alpha,\beta} = \frac{\partial}{\partial u^\beta} x^i_\alpha - \gamma^\rho_{\alpha\beta} x^i_\rho + \Gamma^i_{jk} x^j_\alpha \frac{\partial x^k}{\partial u^\beta}$$

$$= \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} - \gamma^\rho_{\alpha\beta} x^i_\rho + \Gamma^i_{jk} x^j_\alpha x^k_\beta \tag{2.3.15}$$

However, substitution of Eq. (2.2.17) into Eq. (2.3.15) yields

$$x^i_{\alpha,\beta} = 0 \tag{2.3.16}$$

and finally

$$\frac{D}{Dt}(x^i_\alpha) = v^i_{,\alpha} = x^i_\rho v^\rho_{,\alpha} \tag{2.3.17}$$

Now, Eq. (2.3.10) may be expressed in terms of velocities (deformation rates)

$$\frac{D}{Dt} g_{\alpha\beta} = G_{ij} \left(x^i_\rho v^\rho_{,\alpha} x^j_\beta + x^i_\alpha x^j_\rho v^\rho_{,\beta} \right)$$

$$= g_{\rho\beta} v^\rho_{,\alpha} + g_{\alpha\rho} v^\rho_{,\beta} = v_{\alpha,\beta} + v_{\beta,\alpha} \tag{2.3.18}$$

The components of the deformation rate are then, defined as

$$d_{\alpha\beta} \overset{\Delta}{=} \frac{1}{2} \left(v_{\alpha,\beta} + v_{\beta,\alpha}\right) = d_{\beta\alpha} \tag{2.3.19}$$

and thus,

$$\frac{D}{Dt} \left(\log ds\right) = d_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\rho}{ds} \tag{2.3.20}$$

$$\frac{D}{Dt} \left(g_{\alpha\beta}\right) = 2d_{\alpha\beta} \tag{2.3.21}$$

The intrinsic geometer is, by no means, capable of measuring velocities directly, Eq. (2.3.13), yet he can measure metric rates, Eq. (2.3.18), and obtain the velocities by spatial integration of Eq. (2.3.18). Two problems arise in conjunction with this: (a) a system of equations like Eqs. (2.3.19) (i.e., given $d_{\alpha\beta}$ and required v_α is not necessarily compatible, and (b) even if compatibility is ascertained, the solution is not unique. Treatment of the first problem will be postponed, while the second is alleviated by the definition of the spin tensor

$$w_{\alpha\beta} \overset{\Delta}{=} \frac{1}{2} \left(v_{\alpha,\beta} - v_{\beta,\alpha}\right) = - w_{\beta\alpha} \tag{2.3.22}$$

then

$$v_{\alpha,\beta} = d_{\alpha\beta} + w_{\alpha\beta} = d_{\beta\alpha} - w_{\beta\alpha} \tag{2.3.23}$$

The derivatives of the contravariant components $g^{\alpha\beta}$ are readily obtained by the differentiation of Eq. (2.2.9), namely

$$\frac{\partial}{\partial t} \left(g^{\alpha\beta} g_{\beta\gamma}\right) \frac{\partial}{\partial t} \left(\delta^\alpha_\gamma\right) = 0 \tag{2.3.24}$$

hence

$$g_{\beta\gamma} \frac{\partial}{\partial t} g^{\alpha\beta} = -2g^{\alpha\beta} d_{\beta\gamma} \tag{2.3.25}$$

Multiplication by $g^{\beta\gamma}$ and contraction yield

$$\frac{\partial}{\partial t} g^{\rho\alpha} = - 2g^{\alpha\beta}g^{\rho\gamma}d_{\beta\gamma} \tag{2.3.26}$$

As soon as the deformation rate is established as the time derivative of the metric tensor, the intrinsic characteristics of the continuum (i.e., Christoffel symbols, the Riemann-Christoffel tensor, etc.), being metric properties of space, are readily differentiated (with respect to time). This depends on the deformation rate ($d_{\alpha\beta}$) and not on the spin ($w_{\alpha\beta}$). As an illustration, the determinant of the metric tensor (g) is a measure for a volume element (a detailed proof appears in Ref. 49 and the result for the rate of dilitation is

$$\frac{\partial}{\partial t} \sqrt{g} = \sqrt{g}\; g^{\alpha\beta}d_{\alpha\beta} = \sqrt{g}\; v^{\rho}_{,\rho} \tag{2.3.27}$$

It is evident that the spin components do not affect the expressions in Eq. (2.3.27) (dilitation is not affected by the spin).

As Christoffel symbols, Eq. (2.2.16), play an essential role in tensor calculus, the need for an evaluation of their time derivatives cannot be underestimated. Hence

$$\begin{aligned}\frac{\partial}{\partial t} \gamma^{\alpha}_{\rho\sigma} = {} & \frac{1}{2} \left(\frac{\partial}{\partial t} g^{\alpha\beta} \right) \left(\frac{\partial g_{\rho\beta}}{\partial u^{\sigma}} + \frac{\partial g_{\beta\sigma}}{\partial u^{\rho}} - \frac{\partial g_{\rho\sigma}}{\partial u^{\beta}} \right) \\ & + \frac{1}{2} g^{\alpha\beta} \frac{\partial}{\partial t} \left(\frac{\partial g_{\rho\beta}}{\partial u^{\sigma}} + \frac{\partial g_{\rho\beta}}{\partial u^{\sigma}} - \frac{\partial g_{\rho\sigma}}{\partial u^{\beta}} \right)\end{aligned} \tag{2.3.28}$$

Systematic treatment of the first term in Eq. (2.3.28) yields

$$\frac{1}{2} \left(\frac{\partial}{\partial t} g^{\alpha\beta} \right) \cdot 2\gamma^{\mu}_{\rho\sigma}g_{\beta\mu} = - 2\gamma^{\mu}_{\rho\sigma}g^{\alpha\beta}d_{\beta\mu} \tag{2.3.29}$$

Now since the u-coordinates are time-independent, then

$$\frac{1}{2}\ \frac{\partial}{\partial t}\ \left(\frac{\partial g_{\rho\beta}}{\partial u^{\sigma}} \right) = \frac{\partial}{\partial u^{\sigma}}\ \left(\frac{1}{2}\ \frac{\partial g_{\rho\beta}}{\partial t} \right) = \frac{\partial}{\partial u^{\sigma}}\ d_{\rho\beta} \tag{2.3.30}$$

$$= d_{\rho\beta,\sigma} + \gamma^{\mu}_{\rho\sigma}\ d_{\mu\beta} + \gamma^{\mu}_{\beta\sigma} d_{\mu\rho} \quad \text{(cf. Eq. 2.2.18)}$$

Interchanging the triads of indices and rearranging yield

$$\frac{1}{2}\ g^{\alpha\beta}\ \frac{\partial}{\partial t}\ \left(\frac{\partial g_{\rho\beta}}{\partial u^{\sigma}} + \frac{\partial g_{\beta\sigma}}{\partial u^{\rho}} - \frac{\partial g_{\beta\sigma}}{\partial u^{\beta}} \right) = g^{\alpha\beta}\big(d_{\rho\beta,\sigma} + d_{\beta\sigma,\rho} - d_{\rho\sigma,\beta}$$

$$+ 2\ \gamma^{\mu}_{\rho\sigma} d_{\mu\beta}\big) \tag{2.3.31}$$

Substitution of Eqs. (2.3.29) and (2.3.31) into Eq. (2.3.28) yields

$$\frac{\partial}{\partial t}\ \gamma^{\alpha}_{\rho\sigma} = g^{\alpha\beta}\big(d_{\rho\beta,\sigma} + d_{\beta\sigma,\beta} - d_{\rho\sigma,\beta}\big) \tag{2.3.32}$$

In cases where rates like those in Eq. (2.3.32) are desired in terms of velocities, elementary manipulations lead to

$$\frac{\partial}{\partial t}\ \gamma^{\alpha}_{\rho\sigma} = \frac{1}{2}\ \left(v^{\alpha}_{,\rho\sigma} + v^{\alpha}_{,\sigma\rho} + \frac{1}{2}\ v_{\beta}\ \left(r^{\beta\ \ \alpha}_{\cdot\rho\cdot\sigma} + r^{\beta\ \ \alpha}_{\cdot\sigma\cdot\rho} \right) \right. \tag{2.3.33}$$

It should be noticed that although Christoffel symbols are not tensor componets, their time derivatives are. Specialization of Eqs. (2.3.32) and (2.3.33) for Euclidean spaces are self-evident.

The treatment of curvature rates concludes the discussion of the kinematics of the continuum. Regardless of the type of space considered, Eq. (2.2.22) should be differentiated. Then

$$\frac{\partial}{\partial t}\ \left(r^{\alpha}_{\cdot\ \rho\sigma\tau} \right) = \frac{\partial}{\partial t}\ \left(\frac{\partial}{\partial u^{\sigma}}\ \gamma^{\alpha}_{\rho\tau} + \gamma^{\lambda}_{\rho\tau}\gamma^{\alpha}_{\lambda\sigma} - \frac{\partial}{\partial u^{\tau}}\ \gamma^{\alpha}_{\rho\sigma} - \gamma^{\lambda}_{\rho\sigma}\gamma^{\alpha}_{\lambda\tau} \right) \tag{2.3.34}$$

Changing the order of differentiation results in spatial derivatives or a tensor, namely

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \mu^{\sigma}}\gamma^{\alpha}_{\rho\tau}\right) = \frac{\partial}{\partial \mu^{\sigma}}\left(\frac{\partial}{\partial t}\gamma^{\alpha}_{\rho\tau}\right) \tag{2.3.35}$$

Benefitting from the rules of covariant differentiation

$$\frac{\partial}{\partial u^{\sigma}}\left(\frac{\partial}{\partial t}\gamma^{\alpha}_{\rho\tau}\right) = \left(\frac{\partial\gamma^{\alpha}_{\rho\tau}}{\partial t}\right)_{,\sigma} - \gamma^{\alpha}_{\lambda\sigma}\frac{\partial\gamma^{\lambda}_{\rho\tau}}{\partial t}$$

$$+ \gamma^{\lambda}_{\rho\sigma}\frac{\partial\gamma^{\alpha}_{\lambda\tau}}{\partial t} + \gamma^{\lambda}_{\sigma\tau}\frac{\partial\gamma^{\alpha}_{\rho\lambda}}{\partial t}\,, \tag{2.3.36}$$

The first pair of terms in Eq. (2.3.34) reduces to

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial u^{\sigma}}\gamma^{\alpha}_{\rho\tau} + \gamma^{\lambda}_{\rho\tau}\gamma^{\alpha}_{\lambda\sigma}\right) = \left(\frac{\partial\gamma^{\alpha}_{\rho\tau}}{\partial t}\right)_{,\sigma} + \gamma^{\lambda}_{\rho\sigma}\frac{\partial\gamma^{\alpha}_{\lambda\tau}}{\partial t} + \gamma^{\lambda}_{\sigma\tau}\frac{\partial\gamma^{\alpha}_{\rho\lambda}}{\partial t}$$

$$+ \gamma^{\lambda}_{\rho\tau}\frac{\partial\gamma^{\alpha}_{\lambda\sigma}}{\partial t} \tag{2.3.37}$$

The second pair of terms is obtained by changing indices. Substitution into Eq. (2.3.34) and rearrangement yield

$$\frac{\partial}{\partial t}\left(r^{\alpha}_{\cdot\rho\sigma\tau}\right) = \left(\frac{\partial\gamma^{\alpha}_{\rho\sigma}}{\partial t}\right)_{,\sigma} - \left(\frac{\partial\gamma^{\alpha}_{\rho\sigma}}{\partial t}\right)_{,\tau} =$$

$$= g^{\alpha\lambda}\left(d_{\rho\tau,\tau\sigma} - d_{\rho\lambda,\sigma\tau} + d_{\lambda\tau,\rho\sigma} + d_{\rho\tau,\lambda\sigma} - d_{\lambda\sigma,\rho\tau}\right) \tag{2.3.38}$$

As a summary of the kinematic considerations, Eq. (2.3.28) is bivalent. On one hand it relates velocities to the logarithmic rate of deformation. This relation is very useful in applications where velocities are the principal dependent variables. On the other hand, if the deformation rates are given, Eq. (2.3.23) changes its role from formula to equation, and it must be solved for the velocities. In the latter case, Eq. (2.3.38) is an essential asset, because it expresses the compatibility conditions, i.e., a symmetric tensor $d_{\alpha\beta}$ can be introduced into Eq. (2.3.18) to yield the resultant velocities v_α, if they ($d_{\alpha\beta}$) satisty Eq. (2.3.38). The commonplace Euclidean space supplies an immediate manifestation. The the τ's and their time derivatives identically vanish, and the familiar equations of compatibility from the mathematical theory of elasticity (cf. Sokolnikoff Ref. [44]) appear. Kinematics and deformations of surfaces present another example, then an intrinsic relation like Eq. (2.3.38) must be equated to the known time derivatives of the Rieamann-Christoffel tensor (say, from the equation of Gauss, cf. Sokolnikoff[44]).

2.4 Time Derivatives of Tensor Components

Let a continuum in space be moving (translating, rotating, deforming) together with tensor fields associated with its material points. The meaning is that various tensor fields (e.g., forces, stresses, heat fluxes, etc.) are observed and measured in conjunction with material points. This is the Lagrangean viewpoint. However, the position of the observer is very significant. If the observer resides somewhere in the material system it relates the tensor components to a deforming system of coordinates. Thus, the rates by which the components change may appear insufficient for the complete understanding of the observed phenomena. On the other hand, a fixed observer (naturally, in the fixed system), while observing the various fields in conjunction with a specific material point (he is capable of tracing a moving point) associates the components with time-independent metric properties. It should be stressed that the latter observer does not pretend to be Eulerian, it is Lagrangean as the former because its scope of interest (material points rather than spatial points) matters and not his residence.

Hence, let the symbol $\frac{\partial}{\partial t}$ denote the operation of material derivative, by which the rate of change is measured within the material system. Consequently, the tensor components are observed in order to evaluate their rates of change; however, convective terms (cf. Aris[50]) originating from the curvature of the coordinate lines, are also taken into account. On the other hand, the symbol $\frac{d}{dt}$ is used to denote the operation of total derivative, by which, the rate of change is measured from a fixed standpoint. The components, the rates of change of which are evaluated by the operation of the total derivative, are refereed to unchanging coordinate lines. As rates within such a frame of reference seem to be most adequate for several applications of mathematical physics (cf. Sokolnikoff[44], the rules of total differentiation have to be elaborated upon).

Let, then, the transformation rule of Eq. (2.2.13) be rephrased with the aid of the definitions of Eqs. (2.2.23) and (2.2.24), namely

$$A^{i_1 \dots\dots i_p}_{j_1 \dots\dots j_q} = x^{i_1}_{\alpha_1} \dots\dots x^{i_p}_{\alpha_p} u^{\beta_1}_{j_1} \dots\dots u^{\beta_q}_{j_q} a^{\alpha_1 \dots\dots \alpha_p}_{\beta_1 \dots\dots \beta_q} \tag{2.4.1}$$

The total derivative is defined as the rate of change of the components $A^{i_1 \dots\dots i_p}_{j_1 \dots\dots j_q}$ (including the convective effect and those of curved coordinate lines); however, formal implementation of this definition on Eq. (2.4.1) leads to:

$$\frac{d}{dt} A^{i_1 \dots i_p}_{j_1 \dots j_q} = \frac{D}{Dt} \left(x^{i_1}_{\alpha_1} \dots x^{i_p}_{\alpha_p} u^{\beta_1}_{j_1} \dots u^{\beta_q}_{j_q} a^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} \right) \tag{2.4.2}$$

for the fact is that the partial derivatives of all the bracketed terms in Eq. (2.4.2) have already been designed to account for the convective effects. A doubt arises whether the expressions in Eq. (2.4.2) do actually yield a tensor out of a tensor. As a matter of fact. the sole "trouble" is embodied in the partial derivative of the tensor components. Let, then $\bar{u}^{\rho}$ denote another set of material coordinates, and let barred symbols denote quantities in the $\bar{u}^{\alpha}$-system, then, following Eq. (2.2.13)

$$a^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} = \frac{\partial u^{\alpha_1}}{\partial \bar{u}^{\rho_1}} \dots \frac{\partial u^{\alpha_p}}{\partial \bar{u}^{\rho_p}} \frac{\partial \bar{u}^{\sigma_1}}{\partial u^{\beta_1}} \dots \frac{\partial \bar{u}^{\sigma_q}}{\partial u^{\beta_q}} \bar{a}^{\rho_1 \dots \rho_p}_{\sigma_1 \dots \sigma_q} \tag{2.4.3}$$

As a consequence of the partial differentiation

$$\frac{\partial}{\partial t} a^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} = \frac{\partial u^{\alpha_1}}{\partial \bar{u}^{\rho_1}} \dots \frac{\partial u^{\alpha_p}}{\partial \bar{u}^{\rho_p}} \frac{\mu \bar{u}^{\sigma_1}}{\partial u^{\beta_1}} \dots \frac{\partial \bar{u}^{\sigma_q}}{\partial u^{\beta_q}} \frac{\partial}{\partial t} \bar{a}^{\rho_1 \dots \rho_q}_{\sigma_1 \dots \sigma_q}$$

$$= \sum_{\lambda=1}^{p} \frac{\partial u^{\alpha_1}}{\partial \bar{u}^{\rho_1}} \dots \frac{\partial}{\partial t} \left(\frac{\partial u^{\alpha_\lambda}}{\partial \bar{u}^{\rho_\lambda}} \right) \dots \frac{\partial u^{\alpha_p}}{\partial \bar{u}^{\rho_p}} \frac{\partial \bar{u}^{\sigma_1}}{\partial u^{\beta_1}} \dots \frac{\partial \bar{u}^{\sigma_q}}{\partial u^{\beta_q}} \bar{a}^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q}$$

$$= \sum_{\lambda=1}^{q} \frac{\partial u^{\alpha_1}}{\partial \bar{u}^{\rho_1}} \dots \frac{\partial u^{\alpha_p}}{\partial u^{\rho_p}} \frac{\partial \bar{u}^{\sigma_1}}{\partial u^{\beta_1}} \dots \frac{\partial}{\partial t} \left(\frac{\partial u^{\sigma_\lambda}}{\partial u^{\beta_\lambda}} \right) \dots \frac{\partial \bar{u}^{\sigma_q}}{\partial u^{\beta_q}} \bar{a}^{\alpha_1 \dots \alpha_p}_{q_1 \dots \beta_q}$$

$$\tag{2.4.4}$$

and the sums in Eqs. (2.4.4) vanish because the material coordinates are inherently time-independent. Equation (2.4.4) expresses, then, a tensor transformation rule.

Returning to the definition of Eq. (2.4.2), formal operations imply

$$\frac{d}{dt} A^{i_1 \ldots i_p}_{j_1 \ldots j_q} = x^{i_1}_{\alpha_1} \ldots x^{i_p}_{\alpha_p} u^{\beta_1}_{j_1} \ldots u^{\beta_q}_{j_q} \frac{\partial}{\partial t} a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}$$

$$+ \sum_{\lambda=1}^{p} x^{i_1}_{\alpha_1} \ldots \left(\frac{\partial}{\partial t} x^{i_\lambda}_{\alpha_\lambda} \right) \ldots x^{i_p}_{\alpha_p} u^{\beta_1}_{j_1} \ldots u^{\beta_q}_{j_q} a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \tag{2.4.5}$$

$$+ \sum_{\lambda=1}^{p} x^{i_1}_{\alpha_1} \ldots x^{i_p}_{\alpha_p} u^{\beta_1}_{j_1} \ldots \left(\frac{\partial}{\partial t} u^{\beta_\lambda}_{j_\lambda} \right) \ldots u^{\beta_q}_{j_q} a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}$$

The derivatives in the first term of Eq. (2.4.5) are readily obtained from Eq. (2.3.17), those of the second sum may be obtained by differentiation of the identity

$$\frac{D}{Dt} \left(x^i_\alpha \, u^\alpha_j\right) = \frac{D}{Dt} \left(\delta^i_j\right) = 0 \tag{2.4.6}$$

then

$$x^i_\rho v^\rho_\alpha u^\alpha_j + x^i_\alpha \frac{D}{Dt} u^\alpha_j = 0 \tag{2.4.7}$$

Multiplication by u^β_i, contraction and rearranging yield

$$\frac{D}{Dt} u^\beta_j = -v^\beta_{\ ,\rho} \, u^\rho_j \tag{2.4.8}$$

The terms in Eq. (2.4.5) should, then, be elaborated on

$$\frac{\partial}{\partial t} \left(x^{i_\lambda}_{\alpha_\lambda}\right) a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} = x^{i_\lambda}_{\rho} v^\rho_{\ ,\alpha_\lambda} \, a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} = x^{i_\lambda}_{\alpha_\lambda} v^{\alpha_\lambda}_{\ ,\rho} a^{\alpha_1 \ldots \rho \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \tag{2.4.9}$$

$$\frac{\partial}{\partial t}(u^{\beta_\lambda}_{j_\lambda})\, a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = -v^{\beta_\lambda}_{,\rho}\, u^{\rho}_{,j_\lambda}\, a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = -\, u^{\beta_\lambda}_{,j_\lambda}\, v^{\rho}_{,\beta_\lambda}\, a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\rho\ldots\beta_q} \quad (2.4.10)$$

The latter equalities in Eqs. (2.4.9) and (2.4.10) stem from interchanges between repeated indices. Substitution of Eqs. (2.4.9) and (2.4.10) into Eq. (2.4.5) yields

$$\frac{d}{dt} A^{i_1\ldots i_p}_{j_1\ldots j_q} = x^{i_1}_{\alpha_1}\ldots x^{i_p}_{\alpha_p}\, u^{\beta_1}_{j_1}\ldots u^{\beta_q}_{j_q} \Big(\frac{\partial}{\partial t} a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} + \sum_{\lambda=1}^{p} v^{\alpha_\lambda}_{,\rho}\, a^{\alpha_1\ldots\rho\ldots\alpha_p}_{\beta_1\ldots\beta_q} - \sum_{\lambda=1}^{q} v^{\rho}_{,\beta_\lambda}\, a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\rho\ldots\beta_q} \Big) \quad (2.4.11)$$

This is a typical manifestation of the tensor character of the total derivatives. It should then be rephrased as the rule for total differentiation in the material system, namely

$$\frac{d}{dt} a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = \frac{\partial}{\partial t} a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} + \sum_{\lambda=1}^{p} v^{\alpha_\lambda}_{,\rho}\, a^{\alpha_1\ldots\rho\ldots\alpha_p}_{\beta_1\ldots\beta_q} - \sum_{\lambda=1}^{q} v^{\rho}_{,\beta_\lambda}\, a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\rho\ldots\beta_q} \quad (2.4.12)$$

It is obvious that the total derivatives depend on the velocity gradients. Looking forward towards the influences of the various types of motion, Eq. (2.3.23) is utilized to obtain

$$\frac{d}{dt} a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = \frac{d^J}{dt} a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} + \sum_{\lambda=1}^{p} \omega^{\alpha_\lambda}_{\cdot\rho}\, a^{\alpha_1\ldots\rho\ldots\alpha_p}_{\beta_1\ldots\beta_q} - \sum_{\lambda=1}^{q} \omega^{\rho}_{\cdot\beta_\lambda}\, a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\rho\ldots\beta_q} \quad (2.4.13)$$

where the symbol $\frac{d^J}{dt}$ denotes the operation of the Jaumann time derivative (cf. Fung[51]), i.e., total derivative with the rotational effects discarded, namely.

$$\frac{d^J}{dt} a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} = \frac{\partial}{\partial t} a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q} + \sum_{\lambda=1}^{p} d^{\alpha_\lambda}_{\rho}\, a^{\alpha_1\ldots\rho\ldots\alpha_p}_{\beta_1\ldots\beta_q} - \sum_{\lambda=1}^{q} d^{\rho}_{\beta_\lambda}\, a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\rho\ldots\beta_q} \tag{2.4.14}$$

The total time derivative and the Jaumann one have two interesting characteristics: sums and products of tensors are differentiated with the aid of the same rules of the differential calculus, and the derivatvies of the metric tensor vanish identically. The former statement establishes the formal readiness of the time differentiation of tensor equations, while the latter enables lowering (or raising) indices before (or after) the time differentiation without affecting the form of the equations.

Last but not least, the total differentation of the covariant derivatives of tensors should be studied. The exact relation between both differentiations (temporal vs. spatial) is the clue for a reliable rate theory (and applications). Obviously, the covariant derivative of a tensor is a tensor, the covariant rank of which is higher (by one) from that of the original one, hence from Eq. (2.4.12).

$$\begin{aligned}\frac{d}{dt} \left(a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_{q,\rho}}\right) = \frac{\partial}{\partial t} \left(a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_{q,\rho}}\right) + \sum_{\lambda=1}^{p} v^{\alpha_\lambda}_{,\nu}\, a^{\alpha_1\ldots\nu\ldots\alpha_p}_{\beta_1\ldots\beta_{q,\rho}} \\ - \sum_{\lambda=1}^{q} v^{\nu}_{,\beta_\lambda}\, a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\nu\ldots\beta_{q,\rho}} - v^{\nu}_{,\rho}\, a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_{q,\nu}}\end{aligned} \tag{2.4.15}$$

The first term in Eq. (2.4.15) is not familiar, then by Eq. (2.2.27)

$$\frac{\partial}{\partial t} \left(a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_{q,\rho}}\right) = \frac{\partial}{\partial t} \left(\frac{\partial}{\partial u^\rho} a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\right) + \sum_{\lambda=1}^{p} \gamma^{\alpha_\lambda}_{\rho\nu}\, a^{\alpha_1\ldots\nu\ldots\alpha_p}_{\beta_1\ldots\beta_q}$$

$$- \sum_{\lambda=1}^{q} \gamma^{\mu}_{\beta_\lambda \rho} a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \mu \ldots \beta_q}$$

$$= \frac{\partial}{\partial u^\rho} \left(\frac{\partial}{\partial t} a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \right) + \sum_{\lambda=1}^{p} \left(\frac{\partial \gamma^{\alpha_\lambda}_{\rho\mu}}{\partial t} a^{\alpha_1 \ldots \mu \ldots \alpha_p}_{\beta_1 \ldots \beta_q} + \gamma^{\alpha_\lambda}_{\rho\mu} \frac{\partial}{\partial t} a^{\alpha_1 \ldots \mu \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \right)$$

$$- \sum_{\lambda=1}^{q} \left(\frac{\partial \gamma^{\mu}_{\beta_\lambda \rho}}{\partial t} a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots u \ldots \beta_q} + \gamma^{\mu}_{\beta_\lambda \rho} \frac{\partial}{\partial t} a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \mu \ldots \beta_q} \right)$$

$$= \left(\frac{\partial}{\partial t} a^{\alpha_1 \ldots a_p}_{\beta_1 \ldots \beta_q} \right)_{,\rho} + \sum_{\lambda=1}^{p} a^{\alpha_1 \ldots \mu \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \frac{\partial}{\partial t} \gamma^{\alpha_\lambda}_{\rho\mu} \qquad (2.4.16)$$

$$- \sum_{\lambda=1}^{q} a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \mu \ldots \beta_q} \frac{\partial}{\partial t} \gamma^{\mu}_{\beta_\lambda \rho}$$

Now, the first term in Eq. (2.4.16) is obtained by the rule of total differentiation of Eq. (2.4.12), namely

$$\left(\frac{\partial}{\partial t} a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \right)_{,\rho} = \left(\frac{d}{dt} a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \right)_{,\rho} - \sum_{\lambda=1}^{p} \left(v^{\alpha_\lambda}_{,\mu} a^{\alpha_1 \ldots \mu \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \right)_{,\rho}$$

$$+ \sum_{\lambda=1}^{q} \left(v^{\mu}_{,\beta_\lambda} a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \mu \ldots \beta_q} \right)_{,\rho} \qquad (2.4.17)$$

Collecting terms of Eqs. (2.4.15, 2.4.16, 2.4.17) yields

$$\frac{d}{dt} \left(a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_{q,\rho}} \right) = \left(\frac{d}{dt} a^{\alpha_1 \ldots \beta_p}_{\beta_1 \ldots \beta_q} \right)_{,\rho} - v^{\mu}_{,\rho} a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_{q,\mu}}$$

$$+ \sum_{\lambda=1}^{p} a^{\alpha_1 \ldots \mu \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \left(\frac{\partial}{\partial t} \gamma^{\alpha_\lambda}_{\mu\rho} - v^{\alpha_\lambda}_{,\mu\rho} \right) - \sum_{\lambda=1}^{q} a^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \mu \ldots \beta_q} \left(\frac{\partial}{\partial t} \gamma^{\mu}_{\beta_\lambda \rho} - v^{\mu}_{,\beta_\lambda \rho} \right)$$

(2.4.18)

Substitution of Eq. (2.3.33) and utilization of the characteristics of the Riemann-Christoffel tensor (Sokolnikoff[44] presents them as exercises) yield the final result

$$\frac{d}{dt}\left(a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_{q,\rho}}\right) = \left(\frac{d}{dt}\, a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_q}\right)_{,\rho} - v^{\nu}_{,\rho}\, a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_{q,\nu}}$$

$$- \sum_{\lambda=1}^{p} v^{\sigma}\left(r_{\sigma\nu\rho\cdot}{}^{\alpha_\lambda} + r^{\alpha_\lambda}_{\sigma\rho\cdot\nu}\right) a^{\alpha_1\ldots\nu\ldots\alpha_p}_{\beta_1\ldots\beta_q} + \sum_{\lambda=1}^{q} v^{\sigma}\left(r_{\sigma\beta_\lambda\rho\cdot}{}^{\nu} + r^{\nu}_{\sigma\rho\cdot\beta_\lambda}\right) a^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\nu\ldots\beta_q} \qquad (2.4.19)$$

In summary, the operations of time differentiation that of covariant one are permissible only in case of rigid body translations ($v^{\nu} = 0$), and any other motion implies further terms. In a Euclidean space, the effect of higher tensor rank is added. Curved spaces imply more complicated terms originating from compatibility considerations.

2.5 The Principle of the Rate of the Virtual Power

The principle of virtual power (or of virtual velocity), as expressed in Eq. (2.2.35), is equivalent to a set of differential equations (equilibrium) and associated boundary conditions (should be appropriately selected between kinematical and natural). However, practical reasoning concerning solution procedures gives rise to basic doubts, as follows. Let a system of specific body forces (f^j) and specific surface tractions (T^j) act on a solid continuum. It is, natually, deformed and stressed. The stresses satisfy Eqs (2.2.35) or (2.3.34) but their relations with the deformation are not self-evident. There are constitutive theories asserting that considerations of convenience (hyperelasticity) or necessity (hypoelasticity) introduce incremental "stress-strain" relationships, i.e., the stress rate is assumed to depend on the deformation rate. In such cases, equations like Eq. (2.2.35) need be manipulated further in order to obtain terms with stress rates.

The objective of this section is, then, within the scope of obtaining the total time derivative of the principle of virtual power. The resultant virtual theorem may be applicable for the derivation of rate equations (equilibrium rate and incremental boundary conditions) and for the direct development of solution methods (e.g., incremental finite elements). Throughout this section, the space is assumed to be Euclidean and all tensor components, following the notations at the end of Subsection 2.2 (Eqs. (2.2.17) and so forth), are measured only in the material system of coordinates. Hence, the tensor notation reverts to Latin indices, keeping the material system in mind.

Eq. (2.2.35) should, first, be rephrased

$$\int_V \sigma^{ij}\delta v_{j,i}dV - \int_V f^j\delta v_j dm - \int_A T^j\delta v_j dm = 0 \qquad (2.5.1)$$

Comparison between Eqs. (2.5.1) and (2.2.35) may explain the symbol dm, it stands for the mass of a (material) volume element or surface element respectively. The volume and the area are varying during the deformation, the mass remains constant. Total differentiaton of Eq. (2.5.1) yields

$$\frac{d}{dt}\int_V \sigma^{ij}\delta v_{j,i}dV - \frac{d}{dt}\int_V f^j\delta v_j dm - \frac{d}{dt}\int_A T^j\delta v_j dm = 0 \qquad (2.5.2)$$

Partial results are readily obtained

$$\frac{d}{dt}\left[\int_V f^j\delta v_j dm + \int_A T^j\delta v_j dm\right] = \int_V \frac{df^j}{dt}\delta v_j dm + \int_A \frac{dT^j}{dt}\delta v_j dm$$

$$+ \int_V f^j \frac{d}{dt}\frac{\delta v_j}{dt} dm + \int_A T^j \frac{d\delta v_j}{dt} dm \qquad (2.5.3)$$

The first integral in Eq. (2.5.2) needs detailed treatment,

$$\frac{d}{dt}\int_V \sigma^{ij}\delta v_{j,i}dV = \int_V \frac{d\sigma^{ij}}{dt}\delta v_{j,i}dV + \int_V \sigma^{ij}\frac{d(\delta v_{j,i})}{dt}dV + \int_V \sigma^{ij}\delta v_{j,i}\frac{d}{dt}dV \tag{2.5.4}$$

The last term in Eq. (2.5.4) symbolizes the volume changes, however from Eqs. (2.3.21) and (2.3.27) one can obtain

$$\frac{d}{dt}\,dV = d^{K}_{\cdot K}\,dV = v^{k}_{,k}dV \tag{2.5.5}$$

The first term in Eq. (2.5.4) consists of the desired stress rates, and therefore the incremental constitutive equations should be substituted there. The remaining integrals in Eq. (2.5.4) should, then, be elaborated upon, since the time derivative of the (virtual) velocity exists. This is a typical need of the rate of a gradient, by Eq. (2.4.19)

$$\frac{d}{dt}(\delta v_{j,i}) = \frac{d\delta v_j}{dt}_{,i} - v^{k}_{,i}\,\delta v_{j,k} \tag{2.5.6}$$

Collecting terms and rearranging yield

$$\int_V \left(\frac{d\sigma^{ij}}{dt} + \sigma^{ij}v^{k}_{,k} - v^{i}_{,k}\sigma^{kj}\right)\delta v_{j,i}dV - \int_V \rho\,\frac{df^j}{dt}\,\delta v_j dV - \int_A \gamma\,\frac{dT^j}{dt}\,\delta V_j dA$$
$$+\int_V \sigma^{ij}\frac{d\delta v_j}{dt}_{,i}dV - \int_V \rho\, f^j\frac{d\delta v_j}{dt}dV - \int_A \gamma T^j\frac{d\delta v_j}{dt}dA = 0 \tag{2.5.7}$$

At any instant, Eq. (2.5.7) is satisfied. The virtual velocity and its time derivative are, then, independent. Moreover, the last three terms of Eq. (2.5.7) (those depending on the "virtual acceleration") are equivalent to Eq. (2.2.35). Hence, the principle of the rate of virtual power may be obtained in its concise form.

$$\int_V \left(\frac{d\sigma^{ij}}{dt} + \sigma^{ij} v^k_{,k} - \sigma^{kj} v^i_{,k} \right) \delta v_{j,i} dV$$

$$- \int_V \rho \frac{df^j}{dt} \delta v_j dV - \int_A \gamma \frac{dT^j}{dt} \delta v_j dA = 0 \qquad (2.5.8)$$

For further classifications, the total derivative of the stress components will be represented by Jaumann derivative, Eq. (2.4.14), namely

$$\frac{d\sigma^{ij}}{dt} = \frac{d^J\sigma^{ij}}{dt} + \omega^i_{\cdot k}\sigma^{kj} + \omega^j_{\cdot k}\sigma^{ik} \qquad (2.5.9)$$

Let the following integrals be defined

$$I_e = \int_V \frac{d^J\sigma^{ij}}{dt} \delta v_{j,i} dV \qquad (2.5.10)$$

$$I_d = \int_V (\sigma^{ij} d^k_{\cdot k} - \sigma^{kj} d^i_{\cdot k}) \delta v_{j,i} dV \qquad (2.5.11)$$

$$I_r = \int_V \omega^j_{\cdot k}\sigma^{ik} \delta v_{j,i} dV \qquad (2.5.12)$$

Then, substitutions in Eq. (2.5.8) yield

$$I \stackrel{\Delta}{=} I_e + I_d + I_r = \int_V \rho \frac{df^j}{dt} \delta v_j dV + \int_A \gamma \frac{dT^j}{dt} \delta v_j dA \qquad (2.5.13)$$

The integral I_e, Eq. (2.5.10), is expected to be formally similar to the "conventional" integrals of linear elasticity. The integral I_r, Eq. (2.5.12), is the sole outcome of rigid rotations, and the integral I_d, Eq. (2.5.11), is required for completeness.

It is obvious that further manipulations on Eq. (2.5.8), especially the use of Gauss' theorem, lead to the field equations and associated boundary conditions. However, the integral form is quite satisfactory for the utilization of the finite element method. Further inspection of Eq. (2.5.8) should reveal the resemblance and the differences to the integral incremental principles presented by Hibbitt et al[52] and numerous successors. Their scope is quite different (the Piola-Kirchhoff stress tensor) and their formulation contains several simplifications.

2.6 Concluding Remarks

The formulations of this section focus upon tensor rate operators associated with continuum mechanics. Tensor definitions and corollaries are compiled. The deformation rates and related topics regarding the kinematics of space are presented. Compatibility of deformation is elaborated on as an outcome of the curvature of space.

Beside the intrinsic rates of change, total (time) derivatives and partial derivatives of an arbitrary tensor are presented. The material system of coordinates is useful for applications, the spatial is favorable for reference, their correlations are highlighted by the above mentioned rates. Last but not least, the error-prone rate of a gradient is presented and proved to differ from the gradient of a rate.

As a consequence of these formulations, various tensor equations (e.g., equilibrium of elastic solids) can be differential (in time) as all tools are available.

However, the validity of the tensor formulations of this section is restricted. As the time derivatives of the transformation tensor, Eqs. (2.3.11) and (2.3.12), are elaborated upon, the existence of their inverse tensor, Eq. (2.2.24) is assumed. Hence, all cases where the inverse transformation tensor cannot be established are outside the scope of the above mentioned rate operators. These are the cases where the dimensionality of the material coordinates is less than that of the spatial. For instance, unidimensional (arches, beams) or bidimensional (shells, plates) material system of coordinates. Then the various curvature components ought to be considered. The conventional curvatures (namely - radii of tangent or normal circles) have to be accompanied by total curvatures, expressing the intrinsic properties. Rate operators for these values must be taken into account in an orderly fashion.

3. CONSTITUTIVE EQUATIONS

3.1 Introduction

In Section 2, the equations, necessary for the precise treatment of constitutive equations, were presented. The present section contains a development of a three-dimensional theory which describes the incremental (rate) behavior of an elasto-thermo-viscoplastic continuum, in the presence of finite strain. Such a theory forms the necessary foundation for further developments of approximate theories for special structures (beams, plates, shells).

3.2 Incremental Theory of Plasticity

The first formulations of incremental elasto-plastic constitutive equations have been by given St. Venant (1870), Levy (1870), Von Mises (1913), Prandtl (1925) and Reuss (1930). These investigators have formulated various plasticity theories, in a style similar to that of the linear theory of elasticity, and ignored some important points, such as the correct stress rate and the difference between Lagrangenian coordinates and Eulerian coordinates in large deformations. A good historical review and additional details may be found in Refs. 53-54. It should be noted that these works form the basis of what might be called "an engineering theory of plasticity" and many works, books and researches rely upon them.

The recent years brought a need for accurate solutions of elasto-plastic problems with large deformations. It became clear that the classical elasto-plastic theories, mentioned above are not sufficient and it is necessary to give a more exact formulation of the constitutive equations. Three main research directions have been developed: The first direction begins with the works of Hill[17,55]. In this approach the elastic increment, which appears in the constitutive equations, is derived from a finite elastic law. For that purpose a finite elastic energy function (potential) must be defined together with a finite elastic strain.

A second research direction has been suggested by Sedov[38] and independently by Green and Naghdi[56]. The elasto-plastic constitutive equations are based on thermodynamic considerations. The concepts of finite strain and a potential are also used in these works. The formulations are general and have not been applied to particular problems.

The third research direction is not restricted to the theory of plasticity. Some thirty years ago new type of materials have been defined by Truesdell[57-60], the hypoelastic materials. The constitutive equations of these materials relate the stress rate with the strain rate. Some authors have shown that the hypoelastic definition includes, as special cases, various forms of plastic laws[61-64]. Detailed reviews on hypoelasticity are given in Refs. 39 and 65.

The classical incremental theories of plasticity make use of an initial yield condition, a hardening rule, and a flow rule in characterizing the strain-hardening response of a material. Although these classical theories continue to be utilized extensively in finite element computer programs, this may be true only because more suitable models have not yet been developed.

Comparison of the models with experimental results indicates relatively good agreement in uniaxial cases under simple loading conditions. However, for biaxial and triaxial cases and situations where the loading is cyclic, when creep and plasticity interact, and when the strain rates are high, the results are often in disagreement with experiment.

The two most widely used yield conditions are the Tresca (maximum shear stress) and von Mises (J_2 theory) conditions. For isotropic metals, the von Mises yield condition generally provides a better description of initial yielding than does the Tresca condition. However, for rocks and soils, the Tresca condition is often used. Other yield conditions have been proposed, however, these have not found wide use because of their mathematical complexity.

A flow rule is used to separate the total strain increment into elastic and plastic components. The most generally accepted flow rule, termed the normality condition, states that as the stress state of a material point comes into contact with and pierces the material's yield surface, the resulting plastic strain increment is along the outward normal to the yield surface at the point of penetration.

The hardening rule provides a description of the changing size and shape of the subsequent yield surface, during plastic flow. In addition to simple expansion and/or translation, experimental evidence has shown that subsequent yield surfaces may exhibit corners, general distorsion, various Bauschinger effects, and dependence on prior cyclic history, strain rate, temperature and hold time, to mention only a few parameters[66]. For simplicity, most finite element programs make use of hardening rules which account only for expansion and/or translation of the yield surface.

The classical isotropic hardening rule postulates that the yield surface expands uniformly during plastic deformation. In its simplest form wherein one assumes the von Mises yield condition and associated flow rule, the rate of strain hardening may be obtained by relating a value of equivalent total plastic strain to a point on a uniaxial stress-strain curve, so that a simple tensile test is all that is necessary to determine the hardening rule parameters. The simplicity of applying the isotropic hardening rule has made it very popular in finite element plasticity analysis.

In contrast, the kinematic hardening model of Prager-Ziegler[67] proposes that the yield surface translates as a rigid shape during plastic flow; the direction of translation being given by a vector connecting the current center of the yield surface and the current stress state. This gives rise to an ideal Bauschinger effect in which the reverse yield stress is lower by an amount of stress equal to the corresponding prior strain hardening.

The Besseling-White (mechanical sublayer) model[68] makes use of a superposition of elastic-perfectly plastic stress states, in order to approximate strain hardening behavior. This model is often idealized mechanically as a parallel arrangement of elastic-perfectly plastic layers, whose yield stresses are adjusted to duplicate a piece-wise linearization of the uniaxial stress-strain curve (the number of layers being equal to the number of points selected on the stress-strain curve). Like the kinematic model, the mechanical sublayer model predicts a rigid translation of the yield surface.

The hardening model proposed by Mroz[69] employs the concept of a field of surfaces of constant work hardening moduli. Each point of a piece-wise linear uniaxial stress-strain curve is represented in stress space by a surface geometrically similar to the initial yield surface but of different size. The yield surface is assumed to expand and translate within this field, contacting and pushing each surface, along with it, as each is encountered.

Krieg[69] proposed a two surface plasticity model, where the yield surface translates and expands within an enclosing "limit surface", which also is allowed to translate and expand independently of the yield. The hardening modulus is then assumed to be a function of the distance between the two surfaces at the loading point. Another model is the piecewise linear strain hardening theory of Hodge[70], which makes use of a yield polygon.

As experimental evidence points out, isotropic and kinematic hardening tend to bracket actual material response in many cases and, for this reason, a number of combined isotropic-kinematic models have been proposed. Most models are based on a constant ratio of expansion to translation, although some results have been reported for a variable ratio based on accumulated plastic strain[71].

3.3 Fundamental Assumptions

In the treatment of elastic-plastic or elastic-viscoplastic deformations, we have to distinguish between the description as a thermo-mechanical process and the corresponding one by means of thermodynamic state equations. It is sometimes assumed that in the case of process which proceeds through non-equilibrium states, it is fundamentally necessary to start with a description of the process[19,37,72]. Alternatively, it has been proposed that one might assume local equilibrium for the elements of a body and therefore describe the state of the elements, in general, by state equations[73-75]. The consequences of adopting these two approaches become particularly clear when considering the influence of entropy. In the description of the process, entropy is a derived quantity and in principle we can proceed without introducing it. In the description by state equations it is, on the contrary, a necessary state value, which, at least in principle, can be immediately determined. When restricting ourselves to homogeneous, quasi-statical thermomechanical processes, the description by state equations can be viewed as equivalent to that by processes[73,76]. The controversial issues will, thus, not be discussed further.

We are dealing with large, non-isothermal deformation of solid polycrystalline bodies within the frame of classical continuum mechanics and thermodynamics. A phenomenological theory of such coupled thermo-mechanical processes can be based on a material model including four different elements.[77,78]

a) An elastic (or viscoelastic) element representing the reversible thermo-mechanical processes governed by thermodynamical state equations;

b) An element reflecting certain thermo-mechanical processes which lead to changes of the internal material structure, independent of plastic yield and thermally activated creep;

c) A viscoplastic (plastic) element rendering also certain changes of the accompanying constrained equilibrium state; and

d) An element representing thermally activated creep and relaxation phenomena, which also may be connected with corresponding changes of the internal material structure.

Within the frame of the intended phenomenological theory, it is assumed that the respective thermodynamical state of each element is uniquely defined by the actual values of a finite set of external and internal state variables. Moreover, this enables one to introduce an accompanying fictitious state of constrained thermodynamical equilibrium, by means of a fictitious reversible process during which the internal variables are kept constant. This leads to a unique definition of reversible deformations[77,78].

Viscoplastic deformations are attributed to slip processes in certain crystal planes. These slip processes are based essentially on the notion of lattice defects. Roughly, we may distinguish local (bounded) processes restricted to the single grains (but occuring at the same time in all concerned grains) and global processes running through the whole body. The local processes comprehend the generalization and dissolution of lattice defects, the piling up of lattice defects at grain boundaries, etc. They are very sensitive to changes of the stress state, i.e. to the stress increments. The work involved in these local processes is relatively small and mainly non-dissipative. The global processes are mainly dissipative and essentially governed by the actual stress state. The initiation and continuation of global processes, however, is always coupled with local processes.

3.4 Kinematic Considerations

Basic to most of the postulated models of large elastic-plastic deformation behavior is the additive decomposition[79] of d_{rs} and E_{AB} (see Section 2.2.2) into elastic and plastic parts;

$$\overset{E}{d}_{rs} \; d_{rs} = \overset{E}{d}_{rs} + \overset{P}{d}_{rs}, \qquad \dot{E}_{AB} = \overset{E}{\dot{E}}_{AB} + \overset{P}{\dot{E}}_{AB} \tag{3.4.1}$$

The validity of this additive decomposition in the case of finite elastic-plastic strains has been questioned by Lee and his associates[29,80-82] Lee's[29] approach is based on the total purely elastic unloading from the current state to an intermediate unstressed plastically deformed configuration, without any reverse or other kind of plastic flow. The major point in his theory is to decouple the total elastically induced distortion and measure it from a relaxed unstressed state, which is only plastically deformed from the initial to the intermediate configuration. Accordintly, the deformation gradient $\underset{\sim}{F}$ (our transformation tensor in Section 2) is decomposed in the multiplicative form,

$$\underset{\sim}{F} = \overset{E}{\underset{\sim}{F}}\,\overset{P}{\underset{\sim}{F}} \tag{3.4.2}$$

where $\overset{P}{\underset{\sim}{F}}$ transforms a line element from the initial configuration to the intermediate configuration, and $\overset{E}{\underset{\sim}{F}}$ from the latter to the current configuration. The intermediate configuration is chosen in such a way so that $\overset{P}{\underset{\sim}{F}}$ is unaffected by the presence of rigid body motion. The deformation rate tensors, d_{sr} and d_{rs} are then defined. After some manipulations, Lee shows the following relation:

$$d_{rs} = \overset{E}{d}_{rs} + \overset{E}{F}_{rk}\,\overset{P}{d}_{kl}\,\overset{E}{F}{}^{-1}_{ls}\Big|_s + \overset{E}{F}_{rk}\,\overset{P}{W}_{kl}\,\overset{E}{F}{}^{-1}_{ls}\Big|_s \tag{3.4.3}$$

where the subscript s denotes the symmetric parts. Generalization of Lee's theory for anisotropic elasticity was given by Mandel[83].

Lee's theory is based on the assumption that the elastic law does not change with the history of deformation and hence a total elastic unloading can take place. However, it has been shown[84] that after a fair amount of plastic flow has taken place, reverse plastic deformation will result soon upon unloading, even for small strains. Therefore, a total elastic unloading cannot have any physical significance. In view of this, the theory of Lee appears as a special case of the theory of Green and Naghdi[85]. Although not as general as the theory of Green and Naghdi, Lee's theory, on the other hand, has the advantage of being more easily fitted with the physical property of invariance of elasticity with respect to plastic deformation. Mandel[83] in particular has pointed out that the Green-Naghdi theory is not convenient if one wants to include anisotropic elasticity effects. All this can be avoided by the use of the convected coordinates as proposed by Sedov[38] and Lehmann[86]. The formulation presented herein will follow the work of Lehmann.

All quantities from here on will be related to the metric of the coordinate system u^α in the deformed state. Hence,

$$
\begin{aligned}
f^{\alpha}_{\gamma} &= \overset{\circ}{G}{}^{\alpha\beta}\, G_{\beta\gamma} \\
(f^{-1})^{\alpha}_{\gamma} &= G^{\alpha\beta}\, \overset{\circ}{G}_{\beta\gamma}
\end{aligned}
\tag{3.4.4}
$$

where the supercribed $\circ$ relates to the initial state at time $\overset{\circ}{t}$. The deformation rate is,

$$
\begin{aligned}
d^{\alpha}_{\gamma} &= \tfrac{1}{2}\, G^{\alpha\beta}\, \dot{G}_{\beta\gamma} = -\tfrac{1}{2}\, G_{\gamma\beta}\, \dot{G}^{\beta\alpha} \\
&= \tfrac{1}{2}(f^{-1})^{\alpha}_{\beta}\, (\dot{f})^{\beta}_{.\gamma} = -\tfrac{1}{2}(\dot{f}^{-1})^{\alpha}_{.\beta}\, f^{\beta}_{\gamma}
\end{aligned}
\tag{3.4.7}
$$

and the $(\cdot)$ stands for the material derivative.

The deformation gradient may be split into its elastic and its plastic components in the following manner:

$$f^{\alpha}_{\gamma} = \underbrace{G^{\alpha\beta}\, G_{\beta\Gamma}}_{\substack{P \\ F^{\alpha}_{\ .\Gamma}}} \cdot \underbrace{G^{\Gamma\delta} G_{\delta\gamma}}_{\substack{E \\ f^{\Gamma}_{\gamma}}}$$

$$(f^{-1})^{\alpha}_{\gamma} = \underbrace{G^{\alpha\beta}\, G_{\beta\Gamma}}_{\substack{E \\ (f^{-1})^{\alpha}_{\Gamma}}} \cdot \underbrace{G^{\Gamma\delta}\, G_{\delta\gamma}}_{\substack{P \\ (f^{-1})^{\Gamma}_{\ .\gamma}}} \qquad (3.4.6)$$

The use of capital greek subscripts and superscripts ($G_{\beta\Gamma}$) denotes parameters belonging to a fictitious intermediate state, defined by a fictitious reversible process with frozen internal variables, which is in general incompatible. The circumstance of the non-continuous configuration in the unstressed state has been observed by Sedov[38], who points out that convected coordinates, as used herein, become non-Euclidean in this configuration.

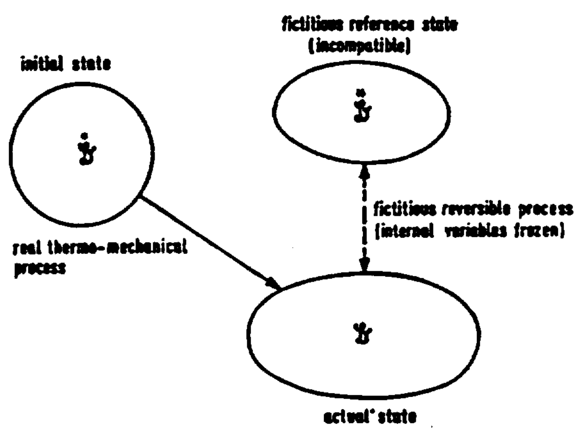

Fig. 3.1 - Definition of Fictious Reference State

This multiplicative splitting of the metric change in the convected coordinates leads to an additive splitting of the deformation rate according to

$$
\begin{aligned}
d^{\alpha}_{\gamma} &= \operatorname{sym} \tfrac{1}{2}\{(\overset{E}{f}{}^{-1})^{\alpha}_{\Gamma}\ (\dot{\overset{E}{f}})^{\Gamma}_{.\gamma}\} + \operatorname{sym} \tfrac{1}{2}\{(f^{-1})^{\alpha}_{\delta}\ (\dot{\overset{P}{f}})^{\delta}_{.\Gamma}\ \overset{E}{f}{}^{\Gamma}_{\gamma}\} = \\
&= \operatorname{sym} \tfrac{1}{2}\ \{(\dot{\overset{E}{f}}{}^{-1})^{\alpha}_{.\Gamma}\ \overset{E}{f}{}^{\Gamma}_{\gamma}\} - \operatorname{sym} \tfrac{1}{2}\{(\overset{E}{f}{}^{-1})^{\alpha}_{\Gamma}\ (\dot{\overset{P}{f}}{}^{-1})^{\Gamma}_{.\delta}\ f^{\delta}_{\gamma}\} = \\
&= \overset{E}{d}{}^{\alpha}_{\gamma} + \overset{P}{d}{}^{\alpha}_{\gamma}
\end{aligned}
\tag{3.4.7}
$$

3.5 Elastic Deformations

The present study is concerned with the structure of the constitutive relation of an elastic-viscoplastic (elastic-plastic) medium. The term elastic-viscoplastic means that the viscosity does not intervene in the elastic domain whose boundary, in particular, is well defined at every stage of the deformation. For simplicity we further assume that the thermo- elastic behavior of the body is isotropic and unaffected by inelastic deformation in the sense that the material constants characterizing the thermo-elastic behavior are independent of inelastic deformation.

It is convenient to work with the Kirchhoff stress tensor, τ, in the current configuration, obtained from the Cauchy stress tensor by scaling

$$\tau^{A}_{B} = \frac{\rho_o}{\rho}\ \sigma^{\alpha}_{\beta} = J\sigma^{\alpha}_{\beta} \tag{3.5.1}$$

where ρ denotes the current mass density, ρ_0 the mass density in the initial state, and J the absolute determinant of the deformation gradient at the current configuration.

The co-rotational stress rate also referred to as Jauman stress rate (see Section 2) will be

$$\begin{aligned} \overset{\nabla}{\sigma}{}^{\alpha}_{\beta} &= \dot{\sigma}^{\alpha}_{.\beta} + d^{\alpha}_{\gamma}\,\sigma^{\gamma}_{\beta} - d^{\gamma}_{\beta}\,\sigma^{\alpha}_{\gamma} \\ \overset{\nabla}{\tau}{}^{\alpha}_{\beta} &= \dot{\tau}^{\alpha}_{\beta} + d^{\alpha}_{\gamma}\,\tau^{\gamma}_{\beta} - d^{\gamma}_{\beta}\,\tau^{\alpha}_{\gamma} \end{aligned} \qquad (3.5.2)$$

From Eq. (3.5.1), the following relations between the various rates of Kirchoff stress and Caushy stress are obtained

$$\begin{aligned} \dot{\tau}^{\alpha}_{\beta} &= \frac{\rho_0}{\rho}\,\dot{\sigma}^{\alpha}_{\beta} + Jd^{\gamma}_{.\gamma}\,\sigma^{\alpha}_{\ \beta} \\ \overset{\nabla}{\tau}{}^{\alpha}_{\beta} &= \frac{\rho_0}{\rho}\,\overset{\nabla}{\sigma}{}^{\alpha}_{\beta} + Jd^{\gamma}_{.\gamma}\,\sigma^{\alpha}_{\beta} \end{aligned} \qquad (3.5.3)$$

If a rate constitutive law is postulated between $\dot{\underset{\sim}{\sigma}}$ and d in finite inelasticity theories, then a potential does not exist, which is necessary in the variational or thermodynamics-based formulation of the problem. The basic difficulty lies with the $d^{\gamma}_{.\gamma}$-term. This is remedied by postulating a constitutive law between $\dot{\underset{\sim}{\tau}}$ and d.

Thus, we can obtain a unique relation between the elastic deformations represented by $\overset{E}{f}{}^{\alpha}_{\gamma}$, the Kirchhoff stresses, τ^{α}_{γ}, and the temperature, T[85,86]:

$$\overset{E}{f}{}^{\alpha}_{\gamma} = \overset{E}{f}{}^{\alpha}_{\gamma}(\tau^{\alpha}_{\gamma}, T) \quad , \quad \tau^{\alpha}_{\gamma} = \tau^{\alpha}_{\gamma}(\overset{E}{f}{}^{\beta}_{\delta}, T), \quad T = T(\tau^{\alpha}_{\gamma}, \overset{E}{f}{}^{\beta}_{\delta}) \tag{3.5.4}$$

This function may be transformed into an incremental relation by differentiation with respect to time. This leads to a general expression of the form

$$\overset{E}{d}{}^{\alpha}_{\gamma} = \overset{E}{d}{}^{\alpha}_{\gamma} \{\tau^{\alpha}_{\gamma}, \overset{\nabla}{\tau}{}^{\alpha}_{\gamma}, T, \dot{T}, G_{\alpha\gamma}, d^{\alpha}_{\gamma}\} \tag{3.5.5}$$

From Eq. (3.5.4), we see that the total deformation rate enters the incremental form of the thermo- elastic stress-strain relations. Therefore, the thermo-elastic deformation is not independent of the inelastic deformation occurring at the same time. This follows from the fact that in the integrated form of the thermo-elastic stress-strain relations, Eq. (3.5.4), the stresses and the strains are referred to the deformed state of the body.

In view of the present discussion and the discussion in the previous subsections, the hyper- elastic behavior described by Eqs. (3.5.4) and (3.5.5) will be replaced by a hypoelastic law. The hypoelastic law is a path-dependent material law, since it cannot be expressed in terms of an initial and a final state; it depends on the path connecting these states. Otherwise, if we did not make such a change, it would be necessary to retain the finite deformation measure in the constitutive law. For small elastic strains, there is practically no difference between hypoelastic and hyperelastic laws, as shown, for example, by Lehmann[86].

The above could be illustrated by the following example. From the frequently used elastic stress-strain relation

$$\varepsilon^{\alpha}_{\gamma} = \tfrac{1}{2}\{\delta^{\alpha}_{\gamma} - (\overset{E}{f}{}^{-1})^{\alpha}_{\gamma}\} =$$

$$= \frac{1}{2G}\{\tau^{\alpha}_{\gamma} - \frac{\nu}{1+\nu}\,\tau^{\beta}_{\beta}\,\delta^{\alpha}_{\gamma}\} + \alpha(T - T_0)\delta^{\alpha}_{\gamma} \tag{3.5.6}$$

we get

$$\overset{E}{d}{}^{\alpha}_{\gamma} = \frac{1}{2G}\left\{\mathrm{sym}[\overset{E}{f}{}^{\delta}_{\gamma}(\dot{\tau})^{\alpha}_{.\delta}] - \frac{\nu}{1+\nu}\,\overset{E}{f}{}^{\alpha}_{\gamma}(\dot{\tau}^{\beta}_{\beta})\right\} + \alpha\dot{T}\,\overset{E}{f}{}^{\alpha}_{\gamma} \tag{3.5.7}$$

which may be replaced by

$$\overset{E}{d}{}^{\alpha}_{\gamma} = \frac{1}{2G}\left\{\overset{\nabla}{\tau}{}^{\alpha}_{\gamma} - \frac{\nu}{1+\nu}\,\overset{\nabla}{\tau}{}^{\beta}_{\beta}\,\delta^{\alpha}_{\gamma}\right\} + \alpha\dot{T}\,\delta^{\alpha}_{\gamma} + \alpha\dot{T}\,\delta^{\alpha}_{\gamma} \tag{3.5.8}$$

We assume that inelastic deformation occurs if and only if

$$F(\tau^{\alpha}_{\gamma},\ T,\ k,\dots,\ \alpha^{\alpha}_{\gamma},\dots,\ A^{\alpha\beta}_{\gamma\delta},\dots) = 0$$

and (3.5.9)

$$\frac{\partial F}{\partial \tau^{\alpha}_{\gamma}}\,\overset{\nabla}{\tau}{}^{\alpha}_{\gamma} + \frac{\partial F}{\partial T}\,\dot{T} > 0$$

for elastic-plastic material,

or

$$F(\tau^{\alpha}_{\gamma},\ T,\ k,\dots,\ \alpha^{\alpha}_{\gamma},\dots,\ A^{\alpha\beta}_{\gamma\delta},\dots) > 0 \tag{3.5.10}$$

for an elastic-viscoplastic material.

The function F represents the yield condition which bounds the domain of pure thermo-elastic behavior, in the ten-dimensional space of stress and temperature. The inequality, given by the second of Eqs. (3.5.9), is the loading condition. The actual form of the yield condition for a given material is determined by a set of so-called internal parameters, which are scalars and/or tensors of even order. The current values of the internal paramters depend on the initial state of the material and the history of the thermo-mechanical process.

3.6 Thermodynamic Considerations

Restricting ourselves to elementary processes, we need not analyze whether the applied heat arises from heat conduction or from heat sources. For the same reason it is not necessary, in our case, to introduce the temperature gradient in addition to the temperature, and the body forces in addition to the stresses.

The first law states, under our simplifying assumptions, that the rate of the specific internal energy, $\dot{U}$, is the sum of the rates of the specific mechanical work, $\dot{W}$, and the specific applied heat, $\dot{q}$:

$$\dot{U} = \dot{W} + \dot{q} \tag{3.6.1}$$

The rate of mechanical work is given by

$$\dot{W} = \frac{1}{\rho_o} \tau^{\alpha}_{\gamma} d^{\gamma}_{\alpha} \tag{3.6.2}$$

and may be split into an elastic and an inelastic part according to Eq. (3.4.7),

$$\dot{W} = \frac{1}{\rho_o} \tau^{\alpha}_{\gamma} \overset{E}{d}{}^{\gamma}_{\alpha} + \frac{1}{\rho_o} \tau^{\alpha}_{\gamma} \overset{P}{d}{}^{\gamma}_{\alpha} = \overset{E}{\dot{W}} + \overset{I}{\dot{W}} \tag{3.6.3}$$

The rate of inelastic work must also be split into a part, $\overset{D}{\dot{W}}$, which is dissipated at once, and into another part $\overset{S}{\dot{W}}$, which represents changes in the internal state. Thus,

$$\overset{I}{\dot{W}} = \frac{1}{\rho_o} \tau^{\alpha}_{\gamma} \overset{P}{d}{}^{\gamma}_{\alpha} = \overset{S}{\dot{W}} + \overset{D}{\dot{W}} \tag{3.6.4}$$

Only $\overset{D}{\dot{W}}$ enters the entropy production

$$T\dot{S} = \dot{q} + \overset{D}{\dot{W}} \tag{3.6.5}$$

The second law of thermodynamics requires

$$\overset{D}{\dot{W}} \geq 0 \tag{3.6.6}$$

We use as thermodynamic state variables the elastic strain, represented by $\overset{E}{f}{}^{\alpha}_{\gamma}$, the absolute temperature T, and a number of other internal state variables $(k,\dots,\alpha^{\alpha}_{\gamma},\dots, A^{\alpha\beta}_{\gamma\delta}\dots)$, which may be scalars and tensors of even order. The choice of $\overset{E}{f}{}^{\alpha}_{\gamma}$ and T as state variables is based on the fact that in pure thermo-elastic deformations, both quantities form a suitable set of thermodynamic state variables. The plastic strain and the total strain are unsuitable as state variables because, in general, they do not uniquely define the state of the material. A conflicting point of view has been expressed in Refs. 87-89. The remaining state variables are added for the sake of the description of the changes of the internal structure of the material.

The specific free energy (Helmholtz function), ϕ , given by

$$\phi = U - Ts \tag{3.6.7}$$

must be a unique function of the thermodynamic state variables

$$\phi = \phi(\overset{E}{f}{}^{\alpha}_{\gamma},\ T,\ k,\ \alpha^{\alpha}_{\gamma},\dots,\ A^{\alpha\beta}_{\alpha\gamma}\dots) \tag{3.6.8}$$

Since the elastic part of the deformation, according to our assumptions, does not depend on the plastic deformation, we may divide the free energy into two different components, as

$$\phi = \overset{E}{\phi}(\overset{E}{f}{}^{\alpha}_{\gamma},\ T) + \overset{S}{\phi}(T,\ k,\dots,\alpha^{\alpha}_{\gamma},\dots,A^{\alpha\beta}_{\gamma\delta},\dots) \tag{3.6.9}$$

where the first component, $\overset{E}{\phi}$,refers to the elastic deformation and the second, $\overset{S}{\phi}$, to the changes of the internal state.

From Eqs. (3.6.1), (3.6.3), (3.6.4), (3.6.5) and (3.6.7) we derive

$$\dot{\phi} = -s\dot{T} + \dot{\overset{E}{W}} + \dot{\overset{S}{W}} \tag{3.6.10}$$

Also, from (3.6.9), we obtain

$$\dot{\phi} = \frac{\partial \overset{E}{\phi}}{\partial f^{\alpha}_{\gamma}} \overset{E}{\overset{\nabla}{f}}{}^{\alpha}_{\gamma} + \frac{\partial(\overset{E}{\phi} + \overset{S}{\phi})}{\partial T} \dot{T}$$

$$+ \frac{\partial \overset{S}{\phi}}{\partial k} \dot{k} + \ldots + \frac{\partial \overset{S}{\phi}}{\partial \alpha^{\alpha}_{\gamma}} \overset{\nabla}{\alpha}{}^{\alpha}_{\gamma} + \ldots + \frac{\partial \overset{S}{\phi}}{\partial A^{\alpha\beta}_{\gamma\delta}} \overset{\nabla}{A}{}^{\alpha\beta}_{\gamma\delta} + \ldots \tag{3.6.11}$$

By comparison of these last two equations, Eqs.(3.6.10) and (3.6.11), we may conclude that

$$s = -\frac{\partial(\overset{E}{\phi} + \overset{S}{\phi})}{\partial T}$$

$$\dot{\overset{S}{W}} = \frac{\partial \overset{S}{\phi}}{\partial k} \dot{k} + \ldots + \frac{\partial \overset{S}{\phi}}{\partial \alpha^{\alpha}_{\gamma}} \overset{\nabla}{\alpha}{}^{\alpha}_{\gamma} + \ldots + \frac{\partial \overset{S}{\phi}}{\partial A^{\alpha\beta}_{\gamma\delta}} \overset{\nabla}{A}{}^{\alpha\beta}_{\gamma\delta} \ldots \tag{3.6.12}$$

$$\tau^{A}_{C} = \rho_{o} \overset{E}{f}{}^{\beta}_{\gamma} \frac{\partial \overset{E}{\phi}}{\partial \overset{E}{f}{}^{\alpha}_{\beta}}$$

For irreversible processes, this scheme of description has to be completed by some statements about the dependence of entropy production on the thermo-mechanical process. Under our assumption we need only deal with entropy production by dissipated mechanical work, in connection with inelastic deformation. Thus, we assume, in general

$$\dot{\overset{D}{W}} = C^{\alpha\beta}_{\gamma\delta}\, \tau^{\gamma}_{\alpha}\, \overset{I}{d}{}^{\delta}_{\beta} > 0 \tag{3.6.13}$$

where

$$C^{\alpha\beta}_{\gamma\delta} = C^{\alpha\beta}_{\gamma\delta}(\overset{E}{f}{}^{\alpha}_{\gamma},\ T,\ k, \ldots \alpha^{\alpha}_{\gamma}, \ldots\ A^{\alpha\beta}_{\gamma\delta}) \tag{3.6.14}$$

Eqs. (3.6.12) and (3.6.13) are the governing equations for non-isothermal, elastic-inelastic elementary processes. The specific free energy ϕ, which determines the non-disspated work of the thermo- mechanical process, and the quantity $C^{\alpha\beta}_{\gamma\delta}$, which governs the entropy production, must be specified according to the material behavior.

3.7 Elastic-Plastic Model

Elementary processes of elastic-plastic bodies may be considered as a sequence of equilibrium states, at least as long as the rate of deformation is moderate, so that the specific free energy is well defined in each state of the processes. We consider a simple example of isotropic behavior. In this case, the state of hardening can be described in the first approximation by a scalar state variable (beside the temperature). In order to simplify more, we assume here that the hardening is independent of temperature. Then the free energy can be written in the form:

$$\phi(\overset{E}{f}{}^{\alpha}_{\gamma},\ T,\ h^2) = \overset{E}{\phi}(\overset{E}{f}{}^{\alpha}_{\gamma}, T) + \overset{S}{\phi}(h^2) \tag{3.7.1}$$

For the work during hardening it follows that:

$$\dot{\overset{S}{W}} = \frac{d\overset{S}{\phi}}{dh^2}\,\dot{h}^2 = \dot{\overset{S}{\phi}}$$

$$\dot{\overset{S}{W}} = \overset{S}{\phi}(h^2) - \overset{S}{\phi}(h_o^2) \tag{3.7.2}$$

Provided now that there exists a uniquely defined relation (depending only on h^2) between the dissipated energy $\overset{D}{W}$ and the work during $\overset{S}{W}$ - i.e.

$$\overset{D}{W} = c(h^2)\,\overset{S}{\phi}\,(h^2) - c(h_o^2)\,\overset{S}{\phi}(h_o^2) \quad , \tag{3.7.3}$$

then we have

$$\begin{aligned} \overset{P}{\dot{W}} &= \overset{S}{\dot{W}} + \overset{D}{\dot{W}} = \frac{d}{dh^2}\left\{[1 + c(h^2)]\,\overset{S}{\phi}(h^2)\right\}\dot{h}^2 \\ \overset{P}{W} &= [1 + c(h^2)]\,\overset{S}{\phi}(h^2) - [1 + c(h_o^2)]\,\overset{S}{\phi}(h_o^2) \end{aligned} \tag{3.7.4}$$

The plastic work, $\overset{P}{W}$, as well as its parts $\overset{S}{W}$ and $\overset{D}{W}$ can, on the assumption made here, be represented as functions of the state variable h^2.

We can introduce a yield condition of the form,

$$F(\tau_\gamma^\alpha,\ k^2) = 0, \tag{3.7.5}$$

where k^2 is here a parameter characterizing the hardening. Since, on the other hand, the boundary of elastic behavior in the stress space can depend only on the state variable k^2, then k^2 is a function of h^2, or

$$k^2 = k^2(h^2) = k^2(\overset{P}{W}) \tag{3.7.6}$$

holds. If the hardening is isotropic, in a more restrictive meaning, the yield condition does not change its shape during plastic deformations, we have to put

$$F(\tau_\gamma^\alpha,\ k^2) = f(\tau_\gamma^\alpha) - k^2 = 0 \tag{3.7.7}$$

Plastic deformation occurs if Eq. (3.7.7) holds and the loading condition is satisfied at the same time - i.e.,

$$\frac{\partial f}{\partial \tau_\gamma^\alpha}\,\overset{\nabla}{\tau}{}_\gamma^\alpha > 0 \tag{3.7.8}$$

We can use the state variable h^2, for instance, by letting $h^2 = \overset{S}{W}$. Then it follows from Eq. (3.7.2) that

$$\overset{S}{\phi}(h^2) = h^2 \tag{3.7.9}$$

Starting from the initial state $h_o^2 = 0$, we get from Eq. (7.3.4),

$$\overset{P}{W} = \left[1 + c(h^2)\right] h^2 \tag{3.7.10}$$

On the other hand, $\overset{P}{W}$ can be represented from the yield condition, Eq. (3.7.7), as a functin of k^2:

$$\overset{P}{W} = \overset{P}{W}(k^2) \tag{3.7.11}$$

From Eqs. (3.7.10) and (3.7.11) we can determine:

$$h^2 = h^2(k^2) \tag{3.7.12}$$

In the case of linear hardening with

$$k^2 = k_o^2 + 2B\,\rho_o\,\overset{P}{W} \tag{3.7.13}$$

we have

$$\overset{P}{W} = \frac{1}{2B\rho_o}\left(k^2 - k_o^2\right) \tag{3.7.14}$$

By putting $c(h^2)$ = const. = c, thus assuming a constant relation between work of hardening and dissipated energy, it finally follows that

$$\overset{S}{\phi}(h^2) = h^2 = \frac{k^2 - k_o^2}{2B\rho_o(1+c)} \tag{3.7.15}$$

For anistropic behavior and again, provided the plastic deformations are independent of temperature, the state equation for the free energy may be assumed approximately i.e. in the following form:

$$\phi = \overset{E}{\phi}\left(f^{\alpha}_{\gamma},\ T\right) + \overset{S}{\phi}\left(h^{2},\ \alpha^{\alpha}_{\gamma},\ A^{\alpha\beta}_{\gamma\delta}\right) \tag{3.7.16}$$

3.8 Elastic-Viscoplastic Model

Thermo-mechanical processes in elastic- viscoplastic bodies cannot be considered as a sequence of equilibrium states, even in the case of the elementary processes, considered here. Elastic-viscoplastic deformations are associated with non-equilibrium states. One consequence of this fact is that we may get a continuation of a process without any change in the independent proces variables. This occurs, for example, in the case of creep with constant stress and temperature or in the case of an adiabatic stress relaxation under constant strain. In such cases, the body moves from a non-equilibrium state to an equilibrium state.

In order to establish the constitutive relations for elastic-viscoplastic bodies, which in the limiting case becoming elastic-inviscidly plastic, we adopt the usual assumption that the stresses, which produce the inelastic deformation, may be expressed as the sum of the so-called athermal or inviscid stresses, $\bar{\tau}^{\alpha}_{\gamma}$, and the viscous overstresses $\overset{*}{\tau}{}^{\alpha}_{\gamma}$:

$$\tau^{\alpha}_{\gamma} = \bar{\tau}^{\alpha}_{\gamma} + \overset{*}{\tau}{}^{\alpha}_{\gamma} = \bar{\tau}^{\alpha}_{\gamma} + \left(\tau^{\alpha}_{\gamma} - \bar{\tau}^{\alpha}_{\gamma}\right) . \tag{3.8.1}$$

This assumption, by no means, detracts from the "unified" concept. The rate-independent limit of visco-plastic constitutive relation was recently discussed by Travnicek and Kratochvil[90]. Hence, the total work rate can be partitioned in the following way:

$$\dot{W} = \dot{\overset{E}{W}} + \underbrace{\dot{\overset{P}{W}} + \dot{\overset{V}{W}}}_{\dot{\overset{I}{W}}} = \frac{1}{\rho_{o}}\,\tau^{\alpha}_{\gamma}\,\overset{E}{d}{}^{\gamma}_{\alpha} + \frac{1}{\rho_{o}}\,\bar{\tau}^{\alpha}_{\gamma}\,\overset{P}{d}{}^{\gamma}_{\alpha} + \frac{1}{\rho_{o}}\,\overset{*}{\tau}{}^{\alpha}_{\gamma}\,\overset{P}{d}{}^{\gamma}_{\alpha} \tag{3.8.2}$$

The viscous part of the work is completely dissipated. Thus, we may write

$$\overset{V}{\dot{W}} = \overset{D_v}{\dot{W}} \tag{3.8.3}$$

Regarding the plastic work, we have already stated that one part is used for changing the internal state and only the remaining part can be considered to be dissipated. Therefore, we must write

$$\overset{P}{\dot{W}} = \overset{S}{\dot{W}} + \overset{D_p}{\dot{W}} \tag{3.8.4}$$

So, we finally obtain

$$\dot{W} = \overset{E}{\dot{W}} + \overset{S}{\dot{W}} + \underbrace{\overset{D_p}{\dot{W}} + \overset{D_v}{\dot{W}}}_{\overset{D}{\dot{W}}} \tag{3.8.5}$$

We have assumed that the changes of the internal state of the material can be regarded as a sequence of equilibrium states. Then, the specific energy is well-defined in each state of the process and we may take the usual overall statement concerning the specific free energy. In so doing, however, we must be aware of the fact that into the part, $\overset{S}{\dot{W}}$, of the plastic work rate, $\overset{P}{\dot{W}}$, only the athermal stresses $\bar{\tau}^{\alpha}_{\gamma}$ enter, since only these stresses are involved in the plastic mechanism. For the same reason, we can only introduce the athermal stresses $\bar{\tau}^{\alpha}_{\gamma}$ into the statement concerning the dissipated plastic work $\overset{D_p}{\dot{W}}$. On the other hand, we have to add the dissipated viscous work $\overset{D_v}{\dot{W}}$ to $\overset{D_p}{\dot{W}}$ in order to obtain the total rate of dissipation. The different mechanisnms for determining the total dissipation and their coupling have been discussed by Perzyna[91].

We now consider an example in which the specific free energy has the following form

$$\phi = \overset{E}{\phi}\left(\overset{E}{f}{}^{\alpha}_{\gamma}, T\right) + \overset{S}{\phi}\left(T, k, \alpha^{\alpha}_{\gamma}\right)$$

$$= \overset{E}{\phi}\left(\overset{E}{f}{}^{\alpha}_{\gamma}, T\right) + k + f(T) + h\alpha^{\alpha}_{\gamma}\,\alpha^{\gamma}_{\alpha} \tag{3.8.5}$$

In this equation h denotes a constant with the dimension of a specific energy like the variable k and the function f(T).

Furthermore, we assume that the dissipation is given by

$$\overset{D_p}{\dot{W}} = \frac{1}{\rho_o}\,\xi\left(\bar{\tau}^{\alpha}_{\gamma} - c\,\rho_o\,h\,\alpha^{\alpha}_{\gamma}\right)\overset{P}{d}{}^{\gamma}_{\alpha}$$

$$\overset{D_v}{\dot{W}} = \frac{1}{\rho_o}\left(\tau^{\alpha}_{\gamma} - \bar{\tau}^{\alpha}_{\gamma}\right)\overset{P}{d}{}^{\gamma}_{\alpha} \tag{3.8.6}$$

where $\xi < 1$ and c denote constants. This leads to

$$\overset{D}{\dot{W}} = \overset{D_p}{\dot{W}} + \overset{D_v}{\dot{W}} = (\xi - 1)\,\overset{P}{\dot{W}} - \xi\,ch\,\alpha^{\alpha}_{\gamma}\,\overset{P}{d}{}^{\gamma}_{\alpha} + \overset{I}{\dot{W}} \tag{3.8.7}$$

Hence, we obtain

$$\overset{S}{\dot{W}} = \overset{I}{\dot{W}} - \overset{D}{\dot{W}} = (1 - \xi)\,\overset{P}{\dot{W}} + \xi\,ch\,\alpha^{\alpha}_{\gamma}\,d^{\gamma}_{\alpha} \tag{3.8.8}$$

On the other hand from Eq. (3.6.12) and (3.8.5) we have

$$\overset{S}{\dot{W}} = \dot{k} + 2h\,\alpha^{\alpha}_{\gamma}\,\overset{\nabla}{\alpha}{}^{\gamma}_{\alpha} \tag{3.8.9}$$

Eqs. (3.8.8) and (3.8.9) are compatible, for instance, if we put

$$\dot{k} = (1 - \xi)\,\overset{P}{\dot{W}} \tag{3.8.10}$$

and

$$\overset{\nabla}{\alpha}{}^{\gamma}_{\alpha} = \tfrac{1}{2}c\;\xi\;\overset{P}{d}{}^{\gamma}_{\alpha} \tag{3.8.11}$$

From Eq. (3.8.10) it follows that, in our case, the plastic work, $\overset{P}{W}$, is equivalent to the thermodynamic state variable k. This is still true if we take ξ as a function of k. But it does not hold in the general case when ξ also depends on the other state variables T and α_γ^α. Eq. (3.8.11) shows that only in a very special case, a very unrealistic one, the state, variables α_γ^α are equivalent to the plastic deformation.

From the thermodynamical considerations, it follows then that we may introduce the quantities k and $\dot{\alpha}_\gamma^\alpha$, defined by Eqs. (3.8.10) and (3.8.11), or any other equivalent set $(\overset{P}{W}, c\rho_0 h\alpha_\gamma^\alpha)$, as internal variables into the corresponding constitutive equations of the process description.

The constitutive equations themselves are not yet determined completely by Eqs. (3.8.5), (3.8.6), (3.8.10) and (3.8.11). These equations only give the restrictive frame for the formulation of the constitutive equations. We may derive a complete set of constitutive equations, which is compatible with this frame, by the following further assumptions:

a) the yield condition is of the form,

$$F = (\bar{t}_\gamma^\alpha - c\,\rho_0\, h\, \alpha_\gamma^\alpha)\,(\bar{t}_\gamma^\alpha - c\,\rho_0\, h\alpha_\gamma^\alpha) - g^2(\overset{P}{W},T) = 0 \tag{3.8.12}$$

where $\bar{t}_\gamma^\alpha$ denotes the deviator of the Kirchhoff stresses $\bar{\tau}_\gamma^\alpha$,

b) the plastic deformation obeys the so-called normality rule,

$$\overset{P}{d}{}_C^A = \dot{\lambda}\,\frac{dF}{d\bar{\tau}_\gamma^\alpha}\ , \tag{3.8.13}$$

c) the relations between the viscous stresses and the inelastic deformation rate are of the form,

$$\overset{P}{d}{}_\gamma^\alpha = \frac{1}{2n}\,\overset{*}{t}{}_\gamma^\alpha = \frac{1}{2n}\,(t_\gamma^\alpha - \bar{t}_\gamma^\alpha) \tag{3.8.14}$$

d) the quantities ξ and c are constant.

We can elimnate the athermal stresses $\bar{\tau}_\gamma^\alpha$ (which are not state variables) from the equations of evolution by considering that the inelastic deformation can be expressed in two different ways. In one, the plastic mechanism is considered, and in the second, the viscous mechanism is considered. From Eq. (3.8.12) we then obtain,

$$\overset{P}{d}{}^{A}_{C} = 2\dot{\lambda}(\bar{t}^{\alpha}_{\gamma} - c\,\rho_o\, h\alpha^{\alpha}_{\gamma}) \tag{3.8.15}$$

while from Eq. (3.8.14) we have

$$\begin{aligned} \overset{P}{d}{}^{\alpha}_{\gamma} &= \frac{1}{2n}\,(t^{\alpha}_{\gamma} - \bar{t}^{\alpha}_{\gamma}) \\ &= \frac{1}{2n}\,\{t^{\alpha}_{\gamma} - c\,\rho_o\, h\alpha^{\alpha}_{\gamma} - (\bar{t}^{\alpha}_{\gamma} - c\,\rho_o\, h\,\alpha^{\alpha}_{\gamma})\} \end{aligned} \tag{3.8.16}$$

By comparing these equations for $\overset{P}{d}{}^{\alpha}_{\gamma}$ we get

$$\dot{\lambda} = \frac{1}{4n}\,\{(\frac{(t^{\alpha}_{\gamma} - c\,\rho_o\, h\alpha^{\alpha}_{\gamma})\,(t^{\alpha}_{\gamma} - c\,\rho_o\, h\alpha^{\alpha}_{\gamma})}{g^2})^{1/2} - 1\} \tag{3.8.17}$$

Following the course of the process in each state, the internal parameters $\overset{P}{\dot{W}}$ and α^{α}_{γ} and therefore also $g^2 = g^2(\overset{P}{W}, \alpha^{\alpha}_{\gamma})$ are known. Thus, we may calculate $\dot{\lambda}$ from Eq. (3.8.17), and then all the other needed quantities such as $\bar{t}^{\alpha}_{\gamma}$ and $\overset{P}{d}{}^{\alpha}_{\gamma}$.

This procedure fails at the point of transition from the elastic domain to elastic-viscoplastic deformation domain, since in this instant $t^{\alpha}_{\gamma} = \bar{t}^{\alpha}_{\gamma}$ and therefore $\dot{\lambda}$ as well as $\overset{P}{d}{}^{\alpha}_{\gamma}$, become zero. But in this case we may calculate $\overset{\overset{P}{\nabla}}{d}{}^{\alpha}_{\gamma}$ from the following considerations: From Eq. (3.8.14) we obtain, because of $\overset{P}{d}{}^{\alpha}_{\gamma} = 0$ and $t^{\alpha}_{\gamma} = \bar{t}^{\alpha}_{\gamma}$,

$$2(t^{\alpha}_{\gamma} - c\rho_o h\, \alpha^{\alpha}_{\gamma})\, \overset{\nabla}{\bar{t}}{}^{\gamma}_{\alpha} - \frac{\partial g^2}{\partial T}\, \dot{T} = 0 \tag{3.8.18}$$

On the other hand, we derive from Eqs. (3.8.14) and (3.8.15), observing that $\dot{\lambda} = 0$ and $t^{\alpha}_{\gamma} = \bar{t}^{\alpha}_{\gamma}$,

$$\overset{\nabla}{\overset{P}{d}}{}^{\alpha}_{\gamma} = 2\ddot{\lambda}(t^{\alpha}_{\gamma} - c\rho_o h\, \alpha^{\alpha}_{\gamma}) = \frac{1}{2n}\, (\overset{\nabla}{t}{}^{\alpha}_{\gamma} - \overset{\nabla}{\bar{t}}{}^{\alpha}_{\gamma}) \tag{3.8.19}$$

From Eqs. (3.8.19), after multiplying by $(t^{\alpha}_{\gamma} - c\rho_o h\, \alpha^{\gamma}_{\alpha})$, we obtain

$$2\ddot{\lambda}\, \bar{g}^2 = \frac{1}{2n}(\overset{\nabla}{t}{}^{\alpha}_{\gamma} - \overset{\nabla}{\bar{t}}{}^{\alpha}_{\gamma})\, (t^{\gamma}_{\alpha} - c\rho_o h\, \alpha^{\gamma}_{\alpha}) \tag{3.8.20}$$

Together with Eq. (3.8.18) this leads to

$$\ddot{\lambda} = \frac{1}{8n\bar{g}^2}\, \{2(t^{\alpha}_{\gamma} - c\rho_o h\, \alpha^{\alpha}_{\gamma})\, \overset{\nabla}{t}{}^{\gamma}_{\alpha} - \frac{\partial \bar{g}^2}{\partial t}\, \dot{T}\} \tag{3.8.21}$$

Having the value of $\ddot{\lambda}$, we may calculate $\overset{\nabla}{d}{}^{\alpha}_{\gamma}$ from Eq. (3.8.19), $\overset{\nabla}{\bar{t}}{}^{\alpha}_{\gamma}$ from Eq. (3.8.19), etc.

3.9 Some Complementary Remarks

Many thermodynamic considerations of non- isothermal, elastic-viscoplastic deformations refer essentially to the general fundamentals, which must be observed in describing such phenomena as thermo-mechanical processes, and then discuss in particular which restrictions follow from the second law of thermodynamics. Only a few papers attempt to describe completely such processes by state equations. Most of these papers introduce plastic strains as thermodynamic state variables. But one may conclude from the consideration of the phenomena in the crystal lattice (dislocations, for example, which have completely passed through the crystal produce plastic strains but no changes of state), as well as from phenomenological observations (different states of hardening can belong to the same plastic strains), that plastic strains in general cannot be regarded as state variables. Furthermore, all these papers consider the plastic work as completely dissipated. This, however, is in contradiction with experimental results, from which it emerges that one part of plastic work is used for producing states of residual stresses in the lattice, which, when phenomenologically considered, cause hardening.

The results, in the work presented here, can be extended to more complex constitutive equations by introducing more internal parameters or state variables, respectively. We may extend our approach to more general, anisotropic hardening materials by assuming (see Eq. (3.8.5)), for example, that

$$\begin{aligned}\phi &= \overset{E}{\phi}\,(\overset{E}{f}{}^{\alpha}_{\gamma}, T) + \overset{S}{\phi}(T, k, \alpha^{\alpha}_{\gamma}, A^{\alpha\beta}_{\gamma\delta}) \\ &= \overset{E}{\phi}(\overset{E}{f}{}^{\alpha}_{\gamma}, T) + k + f(T) + A^{\alpha\beta}_{\gamma\delta}\, \alpha^{\gamma}_{\alpha}\, \alpha^{\delta}_{\beta}.\end{aligned} \tag{3.9.1}$$

Also, it may be more advantageous to replace the assumption in Eq. (3.8.13) for the plastic deformation rate by

$$\overset{P}{d}{}^{\alpha}_{\gamma} = \dot{\lambda}\,\frac{\partial F}{\partial \tau^{\alpha}_{\gamma}} + B^{\alpha\beta}_{\gamma\delta}\,\overset{\nabla}{\tau}{}^{\delta}_{\beta}\;. \tag{3.9.2}$$

This form of this model appears to be more suitable for representing some experimental results in which second order effects and some deviations from the normality rule have been observed. Sometimes the normality rule is considered as a fundamental law based on an entropy production principle. But we should keep in mind that, since not all of the plastic work is dissipated, we cannot expect the total plastic deformation rate to obey the theory of plastic potential, even though the mentioned principles of entropy production are correct.

4. APPLICATION TO PROBLEMS IN EXTENSION AND SHEAR

4.1 Introduction

One of the most challenging aspects of finite strain formulations is to locate an analytical solutions with which to compare a proposed formulation. Typically, as a first problem, a large strain uniaxial test case is analyzed. The uniaxial tensile test is a common and simple way to characterize the stress-strain relation for a given material, since the tensor components used in the constitutive relation will have to be related to this uniaxial test.

In Subsection 4.3 an example how the general constitutive relations developed in Section 3 can be applied to a particular material is shown. This material law is applied for all the other examples.

The case considered in Subsection 4.4 examines the rate-dependent plastic response to a deformation history that includes segments of loading, unloading, and reloading, each occurring at verying strain amplitudes, for a bar. These are surely important problems to be considered; however, they only represent a partial test because the principal stretch directions remain constant. Finally, a problem which was discussed by Nagtegaal and de Jong [92] and others [93] as a problem which demonstrates limitations of the constitutive models in many finite strain formulations is the simple shear problem. This problem is solved as the last example.

4.2 - Uniaxial Irrotational Deformations

The uniaxial tensile test is a common and simple way to characterize the stress-strain relation for a given material. Since the tensor components used in the constitutive relation will have to be related to this uniaxial test, and also to gain a physical understanding of the quantities involved in the analysis, it is both useful and instructive to express the tensor quantities previously discussed in terms of the uniaxial tension test variables. A homogeneous, uniaxial, irrotational deformation will be considered here as a first example.

If the original length of the bar is ℓ_o and its present length is ℓ, then, the transformation between the fix coordinate system and the material coordinate system is

$$x^1 = \frac{\ell}{\ell_o} u^1 \tag{4.2.1}$$

$$u^1 = \frac{\ell_o}{\ell} x^1 \tag{4.2.2}$$

and the metric is

$$g_{11} = g^{11} = \overset{o}{G}_{11} = \overset{o}{G}{}^{11} = 1 \tag{4.2.3}$$

$$G_{11} = \frac{\ell^2}{\ell_o^2} \qquad G^{11} = \frac{\ell_o^2}{\ell^2} \tag{4.2.4}$$

from Eq. (3.4.4) we find

$$f^1_1 = \frac{\ell^2}{\ell_o^2} \tag{4.2.5}$$

and

$$(f^{-1})^1_1 = \frac{\ell_o^2}{\ell^2} \tag{4.2.6}$$

The component of the velocity vector in the fixed system are,

$$v^1 = v_1 = \frac{\dot{\ell}}{\ell_o} \tag{4.2.7}$$

and in the convected material coordinate system they are;

$$v^1 = \frac{\dot{\ell}}{\ell} \qquad v_1 = \frac{\dot{\ell}\,\ell}{\ell_o^2} \tag{4.2.8}$$

The components of the deformation rate in the material system are then,

$$d_{11} = \frac{\ell\,\dot{\ell}}{\ell_o^2} \qquad d^{11} = \frac{\ell_o^2\,\dot{\ell}}{\ell^3} \qquad d^1_1 = \frac{\dot{\ell}}{\ell} \tag{4.2.9}$$

It is worth while to notice that the material rate of a logarithmic strain is equal to the mixed components of the rate of deformation tensor.

The unit normal vectors to the deformed and undeformed areas are one and the same unit vector directed along the bar axis, since the deformation is uniaxial and irrotational. Therefore,

$$\nu^1 = \frac{\ell_o}{\ell} \qquad \nu_1 = \frac{\ell}{\ell_o} \tag{4.2.10}$$

The force transmitted across the cross-sectional area of the bar is dP

$$dP^1 = \frac{\ell_o}{\ell}\,dP \qquad dP_1 = \frac{\ell}{\ell_o}\,dP \tag{4.2.11}$$

The corresponding traction vector components are:

$$T^1 = \frac{P}{\gamma A}\,\frac{\ell_o}{\ell} \qquad T_1 = \frac{P}{\gamma A}\,\frac{\ell}{\ell} \tag{4.2.12}$$

The components of the Cauchy stress tensor in the material system are obtained as:

$$\sigma_1^1 = \frac{P}{A} = \frac{\rho}{\rho_o}\frac{P}{A_o}\frac{\ell}{\ell_o}$$

$$\sigma^{11} = \frac{\ell_o^2}{\ell^2}\frac{P}{A} = \frac{\rho}{\rho_o}\frac{P}{A_o}\frac{\ell_o}{\ell} \tag{4.2.13}$$

$$\sigma_{11} = \frac{\ell^2}{\ell_o^2}\frac{P}{A} = \frac{\rho}{\rho_o}\frac{P}{A_o}\frac{\ell^3}{\ell_o^3}$$

and the components of the Kirchhoff stress tensor are:

$$\tau_1^1 = \frac{\rho_o}{\rho}\frac{P}{A} = \frac{P}{A_o}\frac{\ell}{\ell_o}$$

$$\tau^{11} = \frac{\rho_o}{\rho}\frac{P}{a}\frac{\ell_o^2}{\ell^2} = \frac{P}{A_o}\frac{\ell_o}{\ell} \tag{4.2.14}$$

$$\tau_{11} = \frac{\rho_o}{\rho}\frac{P}{A}\frac{\ell^2}{\ell_o^2} = \frac{P}{A_o}\frac{\ell^3}{\ell_o^3}$$

Observe that the uniaxial component τ_1^1 is the stress actually computed in most uniaxial tension test and is also inaccurately labeled some times as "true stress", since it is usually assumed to be equal to the "true stress" because $\rho \approx \rho_o$ is satisfied almost identically for most metals in the plastic region.

The Jaumann derivative of the components of the Cauchy stress tensor in the material system is:

$$\frac{d^J}{dt}\left(\sigma^1_1\right) \equiv \overset{\nabla}{\sigma}{}^1_1 = \dot{\sigma}^{1\ \cdot}_{\cdot\ 1} + d^1_1\ \sigma^1_1 - \sigma^1_1\ d^1_1$$

$$= \dot{\sigma}^{1\ \cdot}_{\cdot\ 1} = \frac{d}{dt}\left(\frac{P}{A}\right)$$

$$\overset{\nabla}{\sigma}{}^{11} = \dot{\sigma}^{11} + d^1_1\ \sigma^{11} + \sigma^{11}\ d^1_1 = \dot{\sigma}^{11} + 2\ d^1_1\ \sigma^{11}$$

$$= \frac{d}{dt}\left(\frac{P}{A}\ \frac{\ell_o^2}{\ell^2}\right) + 2\ \frac{\dot{\ell}}{\ell}\ \left(\frac{P}{A}\ \frac{\ell_o^2}{\ell^2}\right) = \frac{\ell_o^2}{\ell^2}\ \frac{d}{dt}\ \left(\frac{P}{A}\right)$$

$$\overset{\nabla}{\sigma}{}_{11} = \dot{\sigma}_{11} - d^1_1\ \sigma_{11} - \sigma_{11}\ d^1_1 = \dot{\sigma}_{11} - 2d^1_1\ \sigma_{11}$$

$$= \frac{d}{dt}\ (\frac{P}{A}\ \frac{\ell^2}{\ell_o^2} - 2\ \frac{\dot{\ell}}{\ell}\ \left(\frac{P}{A}\ \frac{\ell^2}{\ell_o^2}\right) = \frac{\ell^2}{\ell_o^2}\ \frac{d}{dt}\ \left(\frac{P}{A}\right)$$

The corotationl (Jauman) rates of the Kirchhoff stress components can be similarly obtained:

$$\overset{\nabla}{\tau}{}^1_1 = \frac{d}{dt}\ \left(\frac{P}{A_o}\ \frac{\ell}{\ell_o}\right)$$

$$\overset{\nabla}{\tau}{}^{11} = \frac{\ell_o^2}{\ell^2}\ \frac{d}{dt}\ \left(\frac{P}{A_o}\ \frac{\ell}{\ell_o}\right)$$

$$\overset{\nabla}{\tau}{}_{11} = \frac{\ell^2}{\ell_o^2}\ \frac{d}{dt}\ \left(\frac{P}{A_o}\ \frac{\ell}{\ell_o}\right)$$

As previously noted in Subsection 2.5, the rate of the internal power can be expressed as a function of the Jauman derivative of Caushy stress and the rate of deformation tensor as in Eqs. (2.5.10) - (2.5.13).

4.3 - Example of Constitutive Relations for an Isotropic Hardening Material

For a carbon steel (45(DIN 1720) in a pure tension test at a moderate temperature and strain rate, we find the material behavior which is shown in Fig. 4.1[94].

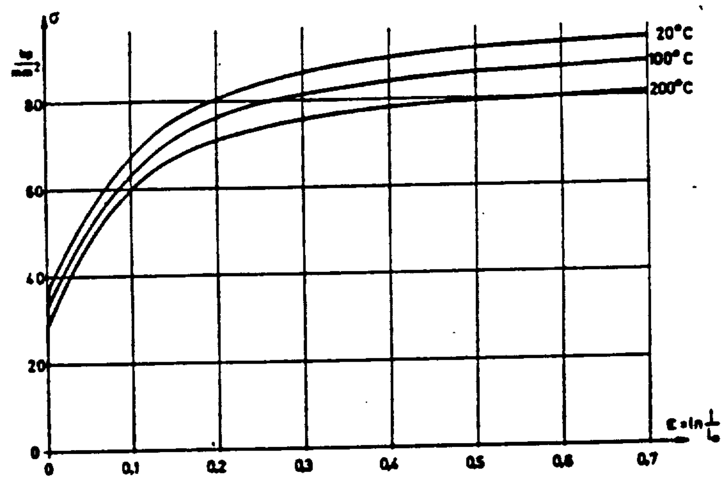

Fig. 4.1 - Carbon Steel C 45 in Tension

From this we may derive the stress-strain-temperature relations for loading in pure tension in the form

$$\sigma = \sigma(\epsilon, T) \tag{4.3.1}$$

For our purpose it is more useful to write this relation in the form

$$\sigma = \sigma(\overset{P}{W}, T) = \frac{c_1(T)\ \overset{P}{W}}{c_2(T) + \overset{P}{W}} + \sigma_o(T) \tag{4.3.2}$$

In our special case we get

$$c_1(T) = 72.42 - 36.03 \cdot 10^{-3} T \ \frac{kp}{mm^2},$$

$$c_2(T) = 7.35 - 8.04 \cdot 10^{-3} T \ \frac{kp}{mm^2}, \tag{4.3.3}$$

$$\sigma_o(T) = 47.41 - 38.9 \cdot 10^{-3} T \ \frac{kp}{mm^2},$$

with T in ^{o}K.

We may consider the carbon steel approximately as an isotropic work-hardening material obeying the v. Misses-Hill yield condition. Furthermore, we assume that a constant ratio of 90% of the plastic work is dissipated. With these assumptions we get the following general process-description for the material under consideration in this example (see Section 3),

independent process variables: τ^{α}_{β}, T

dependent process variables: (a) $\overset{P}{W}$ or $k^2(\overset{P}{W}, T)$, respectively

(b) $f^{\alpha}_{\beta}, \ldots$

yield condition:

$$F(\tau^\alpha_\gamma,\ T,\ \overset{P}{W}) = t^\alpha_\gamma\, t^\gamma_\alpha - k^2(\overset{P}{W},T) = 0, \tag{4.3.4}$$

$$k^2(\overset{P}{W},T) = \frac{2}{3}\left\{\sigma_o(T) + \frac{c_1(T)\ \overset{P}{W}}{c_2(T) + \overset{P}{W}}\right\}^2$$

loading condition:

$$\frac{\partial F}{\partial \tau^\alpha_\gamma}\,\overset{\nabla}{\tau}{}^\alpha_\gamma + \frac{\partial F}{\partial T}\,\dot{T} = 2t^\alpha_\gamma\,\overset{\nabla}{t}{}^\gamma_\alpha - \frac{\partial k^2}{\partial T}\,\dot{T} > 0 \tag{4.3.5}$$

elastic strain rate:

$$\overset{E}{d}{}^\alpha_\gamma = \frac{1}{2G}\left\{\overset{\nabla}{\tau}{}^\alpha_\gamma - \frac{\nu}{1+\nu}\,\overset{\nabla}{\tau}{}^\delta_\delta\,\delta^\alpha_\gamma\right\} + \alpha\dot{T}\,\delta^\alpha_\gamma \tag{4.3.6}$$

plastic strain rate:

when Eqs. (4.3.4) and (4.3.5) are fulfilled:

$$\overset{P}{d}{}^\alpha_\gamma = \dot{\lambda}\,\frac{\partial F}{\partial \tau^\alpha_\gamma} = \frac{2t^\beta_\delta\,\overset{\nabla}{t}{}^\delta_\beta - \dfrac{\partial k^2}{\partial T}\,\dot{T}}{k^2\,\dfrac{\partial k^2}{\partial \overset{P}{W}}} \tag{4.3.7}$$

otherwise:

$$\overset{P}{d}{}^\alpha_\gamma = 0 \tag{4.3.8}$$

rate of plastic work:

$$\dot{\overset{P}{W}} = \tau^\alpha_\gamma\,\overset{P}{d}{}^\gamma_\alpha \tag{4.3.9}$$

rate of applied heat (approximation; the exact formulation is given later in Eq. (4.3.17)):

$$\dot{q} = c\dot{T} - \xi \overset{P}{\dot{W}} \tag{4.3.10}$$

with $\xi = 0.9 = \text{const.}$ $\quad c = 465 \frac{J}{kg\ K}$ (heat capacity)

In the process description the hardening parameter k^2 depends uniquely on the plastic work $\overset{P}{W}$ and the temperature T. This fact leads to the following approach for the corresponding thermodynamics relations (see Subsection 3.6) to make sure that the plastic work is equivalent to thermodynamic state variables:

free energy:

$$\phi = \overset{E}{\phi}(\overset{E}{f}{}^{\alpha}_{\gamma}, T) + h \qquad \text{with } h = \overset{S}{\phi} ; \tag{4.3.11}$$

entropy production:

$$\overset{D}{\dot{W}} = \xi(h)\ \tau^{\alpha}_{\gamma}\ \overset{P}{d}{}^{\gamma}_{\alpha} = \xi(h)\ \overset{P}{\dot{W}} \tag{4.3.12}$$

From this approach and with connection with Eqs. (3.6.4) and (3.6.12), we get

$$\overset{P}{\dot{W}} = \overset{S}{\dot{W}} + \overset{D}{\dot{W}} = \dot{h} + \xi(h)\ \overset{P}{\dot{W}} = \frac{\dot{h}}{1 - \xi(h)} \tag{4.3.13}$$

This means, as required,

$$h = h(\overset{P}{W}) \tag{4.3.14}$$

In our special case, it holds that $\xi(h) = 0.9 = \text{const.}$ Therefore the latent hardening $\overset{S}{\phi} = h$ becomes

$$h = (1-\xi)\ \overset{P}{W} = 0.1\ \overset{P}{W} \tag{4.3.15}$$

From our approach, Eq. (4.3.11), we may also derive for this case, the exact relation between the rate of applied heat, the time derivative of temperature and the deformation rates. Using the Eqs. (3.6.5) and (3.6.12) we get

$$\dot{q} + \overset{D}{\dot{W}} = \dot{q} + \xi \overset{P}{\dot{W}} = T \dot{s} = T\{\frac{\partial s}{\partial \overset{E}{f}{}^{\alpha}_{\gamma}} \overset{\nabla}{f}{}^{\alpha}_{\gamma} + \frac{\partial s}{\partial T} \dot{T} + \frac{\partial s}{\partial h} \dot{h}\}$$

$$= - T\{\frac{\partial^2 \phi}{\partial \overset{\nabla}{f}{}^{\alpha}_{\gamma} \, \partial T} \overset{\nabla}{f}{}^{\alpha}_{\gamma} + \frac{\partial^2 \phi}{\partial T^2} \dot{T} + \frac{\partial^2 \phi}{\partial h \, \partial T} \dot{h}\} \tag{4.3.16}$$

i.e.

$$\dot{q} + \xi \overset{P}{\dot{W}} = - T\{\frac{\partial^2 \overset{E}{\phi}}{\partial T^2} \dot{T} + \frac{\partial^2 \overset{E}{\phi}}{\partial \overset{E}{f}{}^{\alpha}_{\gamma} \, \partial T} \overset{\nabla}{\overset{E}{f}}{}^{\alpha}_{\gamma}\} \tag{4.3.17}$$

In this equation

$$- T \frac{\partial^2 \overset{E}{\phi}}{\partial T^2} = c(\overset{E}{f}{}^{\alpha}_{\gamma}, T) \tag{4.3.18}$$

represent the heat capacity at constant strain ($\overset{E}{d}{}^{\alpha}_{\gamma} = \overset{P}{d}{}^{\alpha}_{\gamma} = 0$). We may consider this heat capacity as constant:

$$c(\overset{E}{f}{}^{\alpha}_{\gamma}, T) = c. \tag{4.3.19}$$

Furthermore, we know from experimental results that the second term on the right-hand side of Eq. (4.3.17) can be neglected in most cases. So we can replace Eq. (4.3.17) by

$$\dot{q} + \xi \dot{W}^{P} = C\dot{T} \qquad (4.3.20)$$

This is identical to the Eq. (4.3.10) which we used as an approximatin in the process description.

4.4 - Uniaxial Cyclic Test

The formulation described in Subsections 4.2 and 4.3 was applied to the characterization of isothermic T = 400°k uniaxial cyclic response to loading program of variable strain amplitudes at a strain rate of 100 S^{-1} (Fig. 4.2(a)).

The dotted lines in Fig. 4.2(b) were obtained under the assumption that the total plastic work was dissipated. Due to the viscid effect, the rate-dependent stress-strain curves (solid lines) have continuous slopes at the shifting points. Note that the transient hardening causes the subcycles not to be closed.

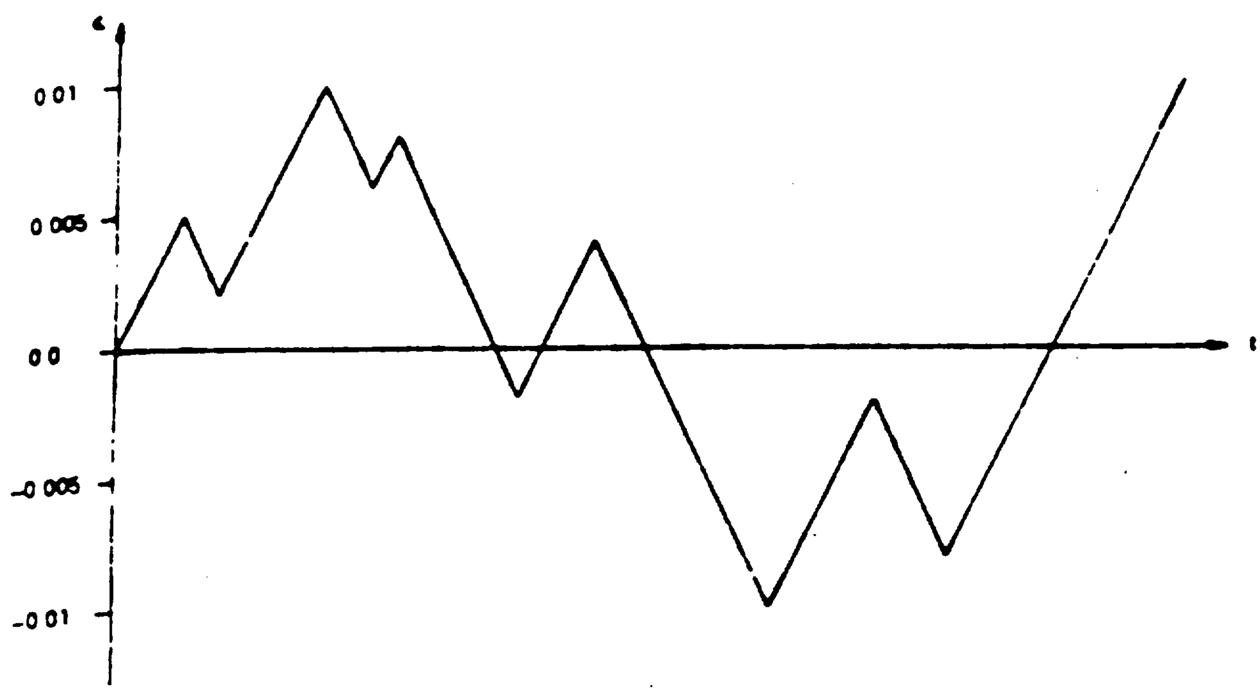

Fig. 4.2(a) - Loading ($\dot{\epsilon}$ = 100 S^{-1})

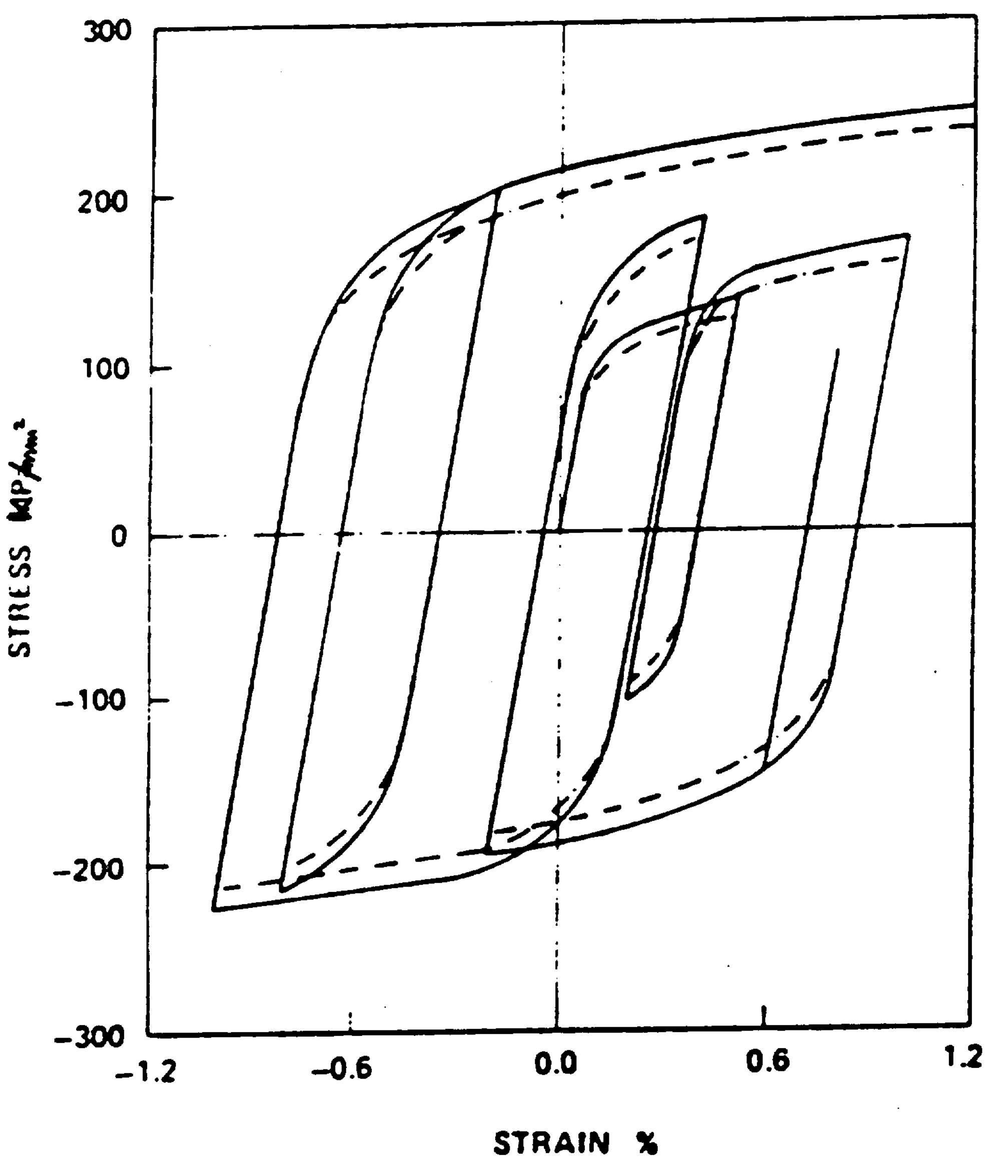

Fig. 4.2(b) - Stress-Strain Response

4.5 - Simple Shear

We take the same material as in subsection 4.3 and consider simple shear process as shown in Fig. 4.3. We denote this material as material A. The processes will

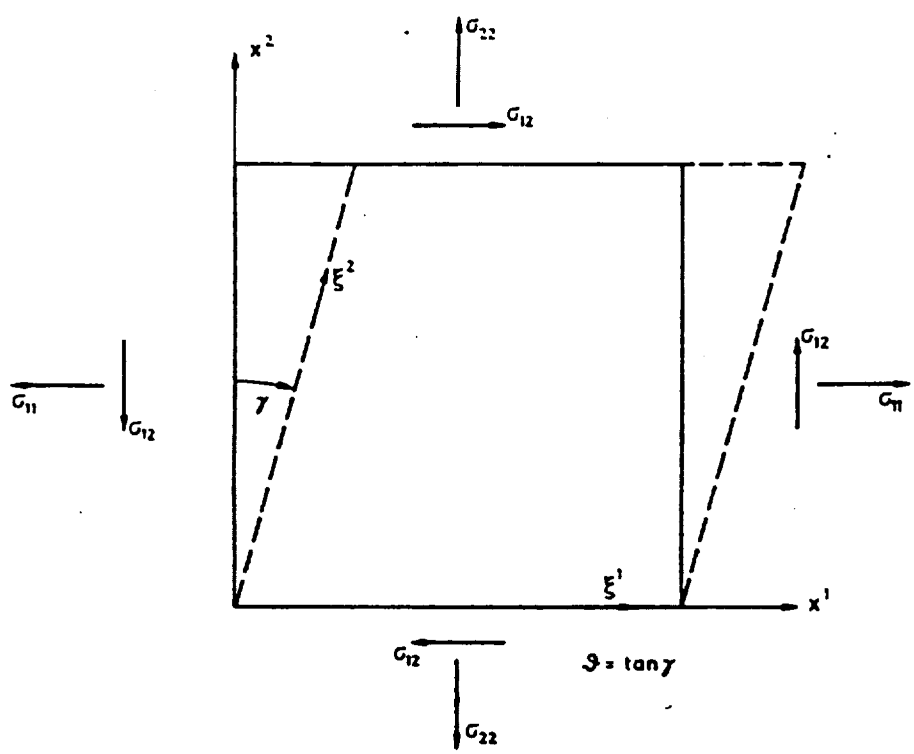

Fig. 4.3 - Simple Shear

be carried out, on the one hand, isothermically and, on the other hand, adiabatically. We find the solution of the problem by numerically integrating a system of first-order differential equations originating from the Eq. (4.3.4) - (4.3.10). In the first case (isothermic process), the total strain rate d^{α}_{γ} and the temperature T_o are given, and in the second case (adiabatic process) the total strain rate and the vanishing of the applied heat are prescribed.

For comparison, we introduce, furthermore, a theoretical material whose yield condition is unaffected by temperature. This means, for this material, the hardening parameter k^2 is

$$k^2 = k^2(\overset{P}{w}, T_o).$$

In isothermic processes this material (denoted as material B) shows the same behavior as material A. But in adiabatic processes we have differences. For material A, the temperature influences the yield condition as well as the elastic part of deformation. For material B, only the elastic components of the deformations are changed by temperature.

Regarding this we must distinguish three cases:

(I): isothermic processes with material A or B,

(IIA): adiabatic processes with material A (hardening rule depending on temperature),

(IIB): adiabatic processes with material B (hardening rule independent of temperature).

The resuls for the shear stresses and the temperature are shown in Fig. 4.4. We see that the differences between the shear stresses in the isothermic and in the adiabatic processes are mainly influenced by the dependence of the yield condition upon temperature. The differences betwen the cases (I) and (IIB) are negligible, but not the differences between IIA and IIB. It should be remarked that in the case (IIA), we get a maximum shear stress for $\nu = 0.87$. So for larger deformation we find in this case a softening effect due to the increasing temperature. With respect to the temperature the differences between the adiabatic cases, (IIA) and (IIB), are rather small, since the differences in the plastic work, in both these cases, are not so important.

The second-order effects are more influenced by the temperature than the first-order effects. This can be see from Fig. 4.5 The effects are partially changed in the opposite direction (see stress σ_{11}). This is due to the strong influence of the temperature on the elastic deformations. We may conclude this from the fact, that the differences between the casses IIA and IIB are less than the differences between I to IIA or IIB, respectively.

From other experiments, however, we know that the observable second-order effects cannot be explained by the influence of the elastic part of the deformation alone. We get more realistic result when we use stress-strain relations derived from the Eq. (3.4.2) with a small correction concerning the theory of plastic potential. In this case the difference between the second-order effects in isothermic and in adiabatic processes may be slightly less. But in any case, the influence of temperature on second-order effects is more important than the influence on first-order effects.

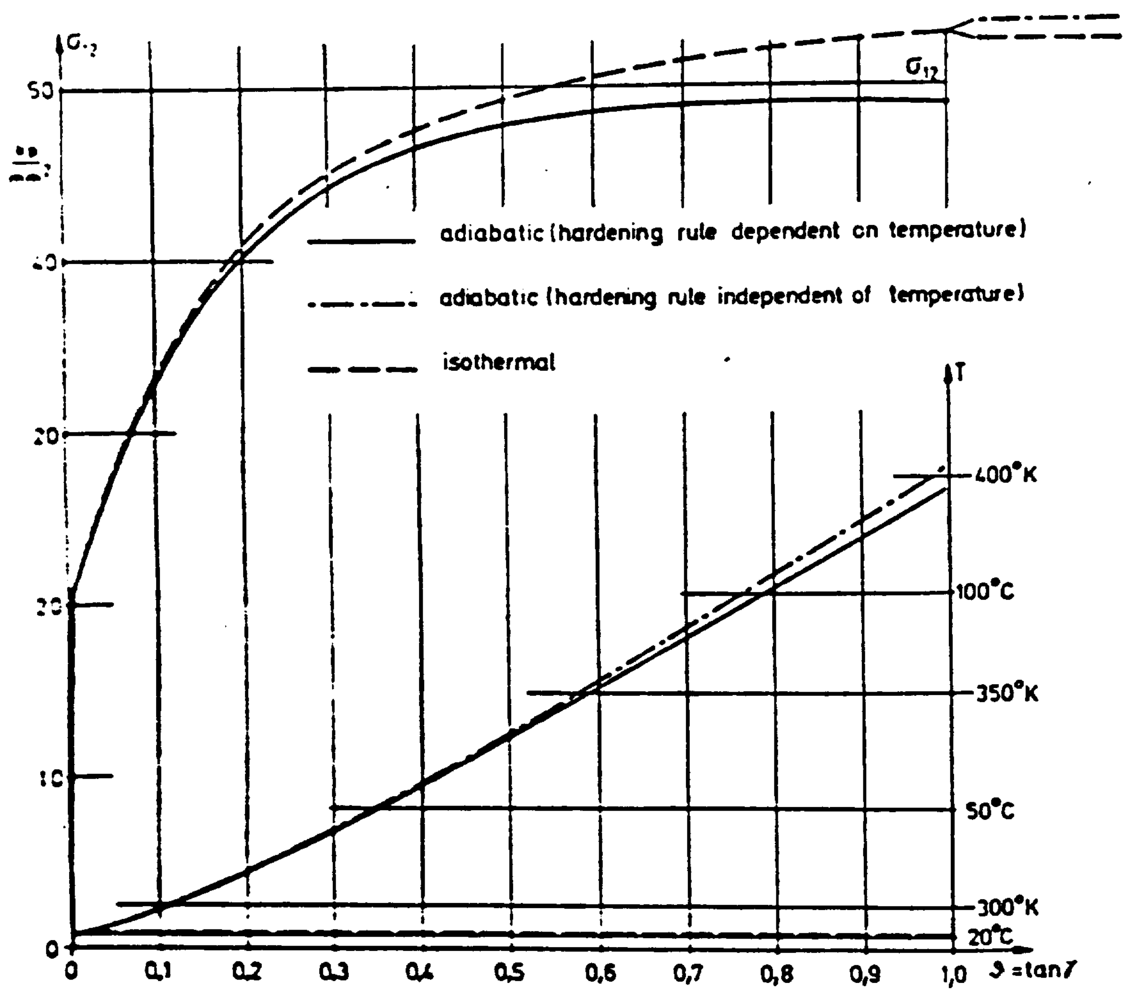

Fig. 4.4 - Response to Simple Shear Test

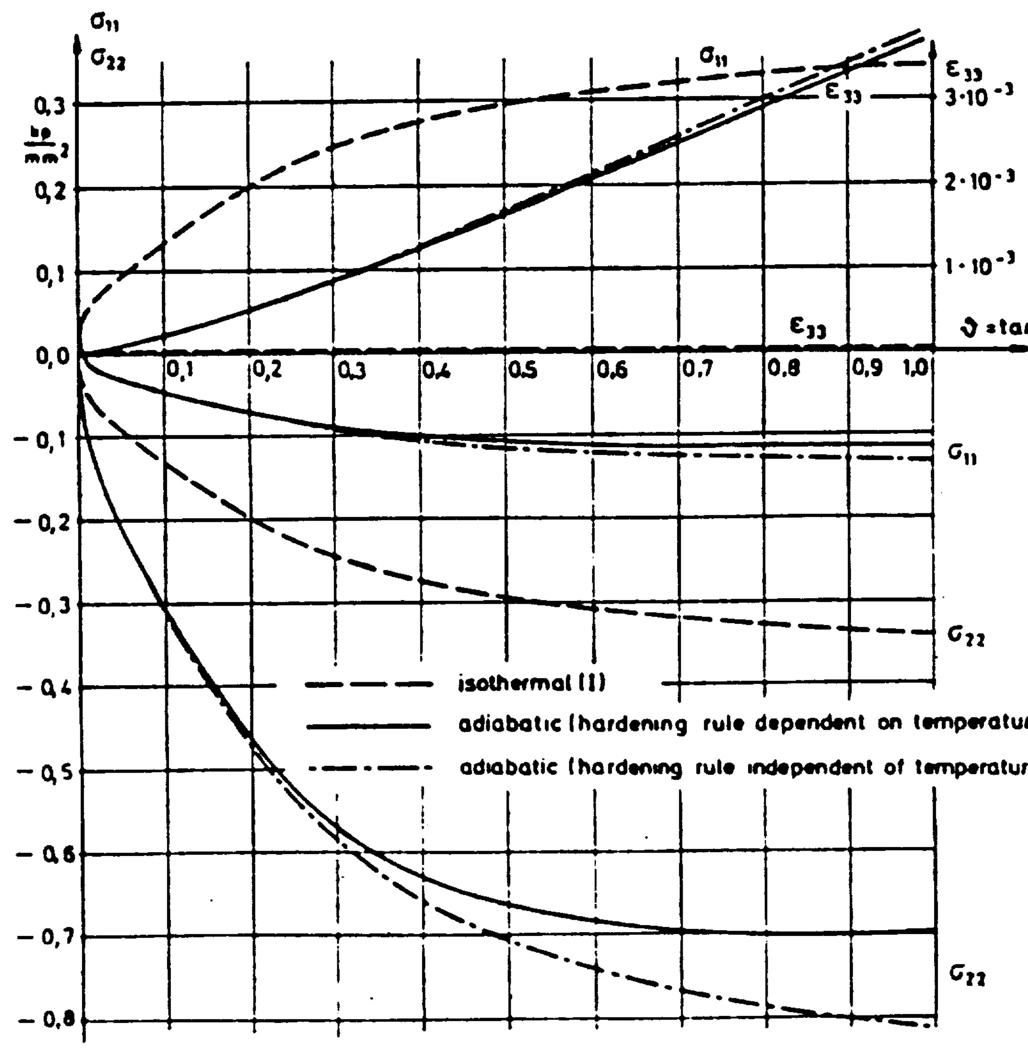

Fig. 4.5 - Response to Simple Shear Test

5. SHELL FORMULATION

5.1 Introduction

A basic framework for nonlinear or buckling analysis of thin shells by the finite element method is supplied by an incrementation of variational principles. Most of the nonlinear theories of shells are Lagrangian in character in that they employ as a reference configuration the undeformed state of the structure. In the construction of these theories it is common practice to start with a set of strain measures and strain-displacement relations (which are usually approximate in some sense), to introduce conjugate stress quantities, and to then postulate a variational principle of virtual work in terms of the variables of the theory. The next step is the incrementation of the principle of virtual work, which introduces further approximations and inconsistencies into the theory.

A new procedure free from all the above cited limitations for the analysis of finite deformations, finite rotations, buckling and post-buckling behavior of arbitrary shaped shells of elastic or inelastic materials is presented in this Section. This formulation is a complete true abinito incremental theory.

Subsection 5.2 describes the geometry and the kinematics of a surface. This formulation is used in the next Subsection which describes the geometry and the kinematics of the shell. The incremental principle of virtual power and incremental equations of equilibrium are introduced in Subsection 5.4.

5.2 Geometry and Kinematics of Surface

The development of differential geometry of surfaces has been presented in numerous books, e.g., McConnel [45] and Sokolnikoff [44]. Following the notation of Ref. [44], a short and summarized development is presented herein.

Surface s imbedded in three dimensional Euclidean space can be defined by:

$$x^i = x^i(u^\alpha) \quad , \quad i = 1,2,3 \quad ; \alpha = 1,2 \tag{5.2.1}$$

where x^i are the coordinates of a point on the surface in the three dimensional system and u^α are the coordinates on surface s.

The covariant components of the metric tensor of the surface (or the first fundamental form of the surface) are:

$$a_{\alpha\beta} = g_{ij}\, x^i_\alpha\, x^j_\beta \tag{5.2.2}$$

The contravariant components, $a_{\alpha\beta}$, of the metric tensor are defined by:

$$a_{\alpha\beta} = a^{\beta\gamma}\, \delta^\gamma_\alpha \tag{5.2.3}$$

A unit vector normal to the surface can be defined as follows:

$$n_i = \frac{1}{2}\, \varepsilon^{\alpha\beta}\, \varepsilon_{ijk}\, x^j_\alpha\, x^k_\beta \tag{5.2.4}$$

where ε_{ijk} and $\varepsilon^{\alpha\beta}$ are the permutation tensors for the space and for the surface.

The total covariant derivative of $a_{\alpha\beta}$ and use of the equality $x^i_{\alpha;\beta} = x^i_{\beta;\alpha}$ bring to the conclusion that:

$$g_{ij}\, x^i_{\alpha;\beta}\, x^j_\gamma = 0 \tag{5.2.5}$$

The implication here is there $x^i_{\alpha;\beta}$ is orthogonal to x^i_γ and consequently to the surface. Thereby, a tensor $b_{\alpha\beta}$ must exist so that:

$$x^i_{\alpha;\beta} = b_{\alpha\beta}\, n^i \tag{5.2.6}$$

Eqs. (5.2.6) are known as the Weingarten equations. With the help of tensor $b_{\alpha\beta}$ one can build the second fundamental form of the surface. The total covariant derivative of the unit normal tensor is obtained by differentiation of the normality conditions:

$$g_{ij}\, x^i_\alpha\, n^j = 0, \tag{5.2.7}$$

yielding

$$n^i_{,\alpha} = -\, b^\gamma_\alpha\, x^i_\gamma \tag{5.2.8}$$

The integrability conditions on x^i are:

$$\frac{\partial^2 x^i}{\partial u^\alpha\, \partial u^\beta} = \frac{\partial^2 x^i}{\partial u^\beta\, \partial u^\alpha} \tag{5.2.9}$$

and they lead (from the covariant derivative of $x^i_{\alpha;\beta}$) to:

$$x^i_{\alpha;\beta\gamma} - x^i_{\alpha;\gamma\beta} = R^\delta_{.\alpha\beta\gamma}\, x^i_\delta\ , \tag{5.2.10}$$

where $R^\delta_{.\alpha\beta\gamma}$ is the Riemamn-Christoffel tensor of the surface given by:

$$R^\delta_{.\alpha\beta\gamma} = \frac{\partial \gamma^\delta_{\alpha\gamma}}{\partial u^\beta} - \frac{\partial \gamma^\delta_{\alpha\beta}}{\partial u^\gamma} + \gamma^\varepsilon_{\alpha\gamma}\, \gamma^\delta_{\varepsilon\beta} - \gamma^\varepsilon_{\alpha\beta}\, \gamma^\delta_{\varepsilon\gamma} \tag{5.2.11}$$

$$R_{\alpha\beta\gamma\delta} = a_{\alpha\varepsilon} R^{\varepsilon}_{.\beta\gamma\delta} \tag{5.2.12}$$

Substitution of Eqs. (5.2.6) and (5.2.8) into Eq. (5.2.10) yields the Gauss-Codazzi equations:

$$b_{\alpha\beta;\gamma} - b_{\alpha\gamma;\beta} = 0 \qquad \text{(Codazzi)} \tag{5.2.13}$$

$$b_{\rho\beta}\, b_{\alpha\gamma} - b_{\rho\gamma}\, b_{\alpha\beta} = R_{\rho\alpha\beta\gamma} \qquad \text{(Gauss)} \tag{5.2.14}$$

The Gaussian curvature of the surface k is defined by:

$$k = \frac{R_{1212}}{a} = \frac{1}{4} R_{\alpha\beta\gamma\delta}\, \varepsilon^{\alpha\beta}\, \varepsilon^{\gamma\delta} \tag{5.2.15}$$

where $a = |a_{\alpha\beta}|$ and $\varepsilon^{\alpha\beta}$ are permutation tensors of the surface ($\varepsilon^{\alpha\beta} = e^{\alpha\beta}/\sqrt{a}$)

or in alternate form:

$$K = \frac{|b_{\alpha\beta}|}{|a_{\alpha\beta}|} \overset{\Delta}{=} \frac{b}{a} \tag{5.2.16}$$

Another important invariant H, the mean curvature of the surface, is defined by:

$$H = \frac{1}{2} a^{\alpha\beta}\, b_{\alpha\beta} \tag{5.2.17}$$

Let us examine for awhile the role of the transformation tensor x^i_α . For any surface contravariant tensor of first order t^α one may write

$$x^i_\alpha\, t^\alpha = T^i \tag{5.2.18}$$

Therefore, the components of t^α are transformed to the space system. A similar transformation for the covariant tensor is by no means obvious because the eq. (5.2.1) is not affine and there is no meaning to write $u^\alpha = u^\alpha(x^i)$. First, Let us define the inverse transformation tensor L_i^α by:

$$L_i^\alpha = g_{ij}\, x_\beta^j\, a^{\alpha\beta} \tag{5.2.19}$$

This definition, Eq. (5.2.19), is obtained by multiplying Eq. (5.2.18) by $g_{ij}\, x_\beta^j$ and by assuming that T^i has only surface components. Then,

$$L_i^\alpha\, T^i = t^\alpha \tag{5.2.20}$$

The inverse transformation tensor L_i^α obeys the following rules:

$$g^{ij}\, L_i^\alpha\, L_j^\beta = a^{\alpha\beta} \tag{5.2.21}$$

$$L_i^\alpha\, x_\beta^i = \delta_\beta^\alpha \tag{5.2.22}$$

$$L_i^\alpha\, n^i = 0 \tag{5.2.23}$$

Moreover, it can easily be verified that:

$$L_{i;\beta}^\alpha = b_\beta^\alpha\, n_i \tag{5.2.24}$$

$$b_\beta^\alpha = -\, L_i^\alpha\, n_{;\beta}^i \tag{5.2.25}$$

The problems with the transformations, depicted by Eqs. (5.2.20) and (5.2.18), stems from the limitation on tensor T^i to have only surface components. From a general tensor T^i only the surface components are obtained by those transformations (in the case of higher rank tensor a little bit different component are lost). In order to overcome this problem, we define the surface system slightly different and add a third dimension by a unit vector normal to the surface. This system will be called latter on τ-system. The transformation tensors for the new system are defined as follows:

$$x^i_\Gamma = \begin{cases} x^i_\alpha & ; \quad \Gamma = 1,2 \\ n^i & ; \quad \Gamma = 3 \end{cases} \tag{5.2.23}$$

and accordingly,

$$L^\Gamma_i = \begin{cases} L^\alpha_i & ; \quad \Gamma = 1,2 \\ n_i & ; \quad \Gamma = 3 \end{cases} \tag{5.2.24}$$

The components of the covariant metric tensor for the new τ-system are

$$a_{\Gamma\Lambda} = g_{ij}\, x^i_\Gamma\, x^j_\Lambda = a_{\alpha\beta}\, \delta^\alpha_\Gamma\, \delta^\beta_\Lambda + n^i\, n_i\, \delta^3_\Gamma\, \delta^3_\Lambda \tag{5.2.25}$$

Similarly, the components of the contravariant metric tensor are:

$$a^{\Gamma\Lambda} = g^{ij}\, L^\Gamma_i\, L^\Lambda_j = a^{\alpha\beta}\, \delta^\Gamma_\alpha\, \delta^\Lambda_\beta + n_i\, n^i\, \delta^\Gamma_3\, \delta^\Lambda_3 \tag{5.2.26}$$

Now, the formal transformation of any tensor from the space system to the new τ-system can be accomplished by:

$$T^i = x^i_\Gamma\, t^\Gamma = x^i_\alpha\, T^\alpha + n^i T^3 \tag{5.2.27}$$

or

$$T_i = L^\Gamma_i\, t_\Gamma = L^\alpha_i\, t_\alpha + n_i\, t_3 \tag{5.2.28}$$

This system, just defined, will be employed in the discussion of shell structures.

Let us assume now that surface s is deformed in time so that the space coordinates of a point on the surface are time dependent. Hence instead of Eq. (5.2.1), we may now write

$$x^i = x^i(u^\alpha, t) \tag{5.2.29}$$

According to the three dimensional formulation in section two, the x^i system is the system at rest and the u^α system is the material system (u^α being independent of t). At this stage of the formulation we need to obtain the time derivatives of all the static parameters defined earlier in this section.

The rate of change of the transformation tensor x^q_α is paramount in the description of the deformation of the surface. Hence repeating the results of section 2.3 [Eq. (2.3.12)];

$$\frac{Dx^q_\alpha}{Dt} = \frac{\partial}{\partial u^\alpha}(V^q) + \Gamma^q_{sk} V^k x^s_\alpha = V^q_{;\alpha} \overset{\Delta}{=} V^q_{,\alpha} \tag{5.2.30}$$

where the velocity vector in the fixed system is given by Eq. (2.3.6), or

$$V^i = \frac{dx^i}{dt} = \frac{\partial x^i}{\partial t} = \frac{Dx^i}{Dt} \tag{5.2.31}$$

However, the components of the spatial velocity vector can be rephrased as follows,

$$V^i = x^i_\gamma V^\gamma + n^i N \tag{5.2.32}$$

From this the expression, $V^q_{;\alpha}$ is obtained through covariant differentiation

$$V^i_{;\alpha} = x^i_\gamma(V^\gamma_{;\alpha} - b^\gamma_\alpha W) + n^i(b_{\gamma\alpha} V^\gamma + W_{;\alpha}) \tag{5.2.33}$$

Comparison of Eqs. (5.2.33) and (5.2.27) is suggesting the possibility of simplifying the formulation by using the τ-system. To accomplish this, there is a need to formulate an additional parameter, the material derivative of the unit vector normal to the surface, n^i. By time differentiation of $g_{ij}\, n^i\, n^j = 1$, one can come to the conclusion that the material time-derivative of n^i, $\frac{Dn^i}{Dt}$ is orthogonal to n^i itself and located on a plane which is tangential to the surface. Hence, one can represent $\frac{Dn^i}{Dt}$ by a linear combination of x^j_α, or

$$\frac{Dn^j}{Dt} = C^\alpha x^j_\alpha \tag{5.2.34}$$

Differentiation of the orthogonality condition, $g_{ij}\, x^i_\alpha\, n^j = 0$, and use of Eqs. (5.2.30) and (5.2.34) yields the following expression,

$$g_{ij}\, V^i_{,\alpha}\, n^j + g_{ij}\, x^i_\alpha\, C^\gamma\, x^j_\gamma = 0 \tag{5.2.35}$$

Rearranging the terms of and substitution of Eq. (5.2.33) into Eq. (5.2.34) yield the following expression for $\frac{Dn^j}{Dt}$,

$$\frac{Dn^j}{Dt} = -\, a^{\alpha\gamma}[b_{\alpha\beta}\, V^\beta + W_{;\alpha}]\, x^j_\gamma \tag{5.2.36}$$

In order to shorten the formulation we introduce two new parameters,

$$\psi^\gamma_{.\alpha} \overset{\Delta}{=} V^\gamma_{,\alpha} - b^\gamma_\alpha\, W = a^{\gamma\delta}\, \psi_{\delta\alpha} \quad a^{\gamma\delta}[V\delta_{;\alpha} - b_{\delta\alpha}\, W] \tag{5.2.37}$$

$$\rho_\alpha \overset{\Delta}{=} b_{\gamma\alpha}\, V^\gamma + W_{;\alpha} = a_{\alpha\delta}\rho^\delta = a_{\alpha\delta}[b^\alpha_\gamma\, V^\gamma + a^{\alpha\gamma}\, W_{;\gamma}] \tag{5.2.38}$$

Next, the different tensor will be denoted in the τ-system. The components of the velocity vector, Eq. (5.2.32), can be written as,

$$V^i = x^i_\Gamma\, V^\Gamma \tag{5.2.39}$$

Eq (5.2.30), can be extended to:

$$\frac{Dx^i_\Gamma}{Dt} = V^i_{;\Gamma} = V^i_{,\alpha}\ \delta^\alpha_\Gamma + \frac{Dn^i}{Dt}\ \delta^3_\Gamma \tag{5.2.40}$$

The second term on the right hand of Eq. (5.2.40) stems from the definition of x^q_Γ [Eq. (5.2.23)]. Note that by employing Eqs. (5.2.33), (5.2.37) and (5.2.38); one can write

$$V^i_{;\Gamma} = [x^i_\gamma\ \psi^\gamma_{.\alpha} + n^i\ \rho_\alpha]\ \delta^\alpha_\Gamma - [x^i_\gamma\ \rho^\gamma]\delta^3_\Gamma \tag{5.2.41}$$

It is useful for the forthcoming development to calculate the material derivatives of the inverse transformation tensor L^α_i (or L^Γ_i in the τ-system). By definition through Eq. (5.2.19), and in the τ-system, Eq. (5.2.24), time differentiation yields

$$\frac{DL^\Gamma_i}{Dt} = \frac{DL^\alpha_i}{Dt} + \frac{Dn_i}{Dt}\ \delta^\Gamma_3$$

$$= [-L^\gamma_i\ \psi^\alpha_{.\gamma} + n_i\ \rho^\alpha]\ \delta^\Gamma_\alpha - [L^\gamma_i\ \rho^\gamma]\ \delta^\Gamma_3 \tag{5.2.42}$$

Consideration of the structure of Eqs. (5.2.41) and (5.2.42) suggests the possibility of simplifying the notation considerably by introducing a new tensor $\psi_{\Gamma\Lambda}$, in the τ-system, in the following way:

$$\psi_{\Gamma\Lambda} = g_{ij}\ x^i_\Gamma\ \frac{Dx^i_\Lambda}{Dt} \tag{5.2.43}$$

A detailed derivation of Eq. (5.2.43), taking into consideration the defintions given by eqs. (5.2.37) and (5.2.38), shows that,

$$\psi_{\Gamma\Lambda} = \psi_{\alpha\beta}\delta^\alpha_\Gamma\ \delta^\beta_\Lambda + (-\rho_\alpha)\ \delta^\alpha_\Gamma\ \delta^3_\Lambda + \rho_\beta\ \delta^3_\Gamma\ \delta^\beta_\Lambda + 0\cdot\delta^3_\Gamma\ \delta^3_\Lambda \tag{5.2.44}$$

Hence:

$$\psi_{\Gamma 3} = - \psi_{3\Gamma} = - \rho_\alpha \delta^\alpha_\Gamma \tag{5.2.45}$$

and

$$\psi_{33} = 0 \tag{5.2.46}$$

Now, we may write Eqs. (5.2.41) and (5.2.42) as follows,

$$\frac{Dx^i_\Gamma}{Dt} = V^i_{;\,\Gamma} = x^i_\Lambda \psi^{\Lambda}_{.\Gamma} \tag{5.2.47}$$

and

$$\frac{DL^\Gamma_i}{Dt} = - L^\Lambda_i \psi^\Gamma_{.\Lambda} \tag{5.2.48}$$

In view of these simple expressions, Eqs. (5.2.47) and (5.2.48), one can appreciate the importance of the introduction of the τ-system. It is worth noticing, at this point, that, although most of the tensor quantities will be expressed in τ-system, the mathemtical operations, like the material derivative, will be performed in the material system u^α . This is done because the τ-system is not a true material system, because of the third direction which is taken to be normal to the surface.

The next step is to calculate the material derivative of the metric tensor in the τ-system, Eq. (5.2.25):

$$\frac{Da_{\Gamma\Lambda}}{Dt} = \frac{D}{Dt} [g_{ij} \, x^i_\Gamma \, x^i_\Lambda] \tag{5.2.49}$$

Since g_{ij} is independent of time, we have

$$\frac{Da_{\Gamma\Lambda}}{Dt} = g_{ij} \left[\frac{Dx^i_\Gamma}{Dt} x^j_\Lambda + x^i_\Gamma \frac{Dx^j_\Lambda}{Dt} \right] \tag{5.2.50}$$

But from the definition of $\psi_{\Gamma\Lambda}$ [Eq. (5.2.43)] and from Eqs. (5.2.47) and (5.2.25) we obtain:

$$\frac{Da_{\Gamma\Lambda}}{Dt} = [\psi_{\Gamma\Lambda} + \psi_{\Lambda\Gamma}] \tag{5.2.51}$$

It is clear from its definition, Eqs. (5.2.43), that $\psi_{\Gamma\Lambda}$ is not a symmetric tensor. The description of the components of $\psi_{\Gamma\Lambda}$ in the u^α system is

$$\frac{Da_{\Gamma\Lambda}}{Dt} = (V_{\alpha;\beta} + V_{\beta;\alpha} - 2Wb_{\alpha\beta})\, \delta^\alpha_\Gamma\, \delta^\beta_\Lambda \tag{5.2.52}$$

and the remaining of the (derivative) components, which permit $\Gamma = 3$ or $\Lambda = 3$ or $\Gamma = \Lambda = 3$, vanish.

The material derivative of the contravariant components of the metric tensor is obtained in a similar manner from Eqs. (5.2.26) and (5.2.48);

$$\begin{aligned} \frac{Da^{\Gamma\Lambda}}{Dt} &= - a^{\Gamma\Omega}[\psi^{\Lambda}{}_{.\Omega} + \psi_{\Omega}^{.\Lambda}] = \\ &= - a^{\Gamma\Omega}\, a^{\Lambda\phi}[\psi_{\phi\Omega} + \psi_{\Omega\phi}] = \\ &= - a^{\alpha\rho}\, a^{\gamma\beta}(v_{\rho;\gamma} + v_{\gamma;\rho} - 2Wb_{\gamma\rho})\, \delta^\Gamma_\alpha\, \delta^\Lambda_\beta \end{aligned} \tag{5.2.53}$$

The derivative of the metric tensor determinant $a = \det(a^{\alpha\beta})$, in the u^α system is obtained by:

$$\frac{Da}{Dt} \overset{\Delta}{=} \frac{\partial a}{\partial t} = \frac{\partial a}{\partial a_{\alpha\beta}} \cdot \frac{\partial a_{\alpha\beta}}{\partial t} = \frac{\partial a}{\partial a_{\alpha\beta}} \cdot \frac{Da_{\alpha\beta}}{Dt} \tag{5.2.54}$$

where it is recognized that $a_{\alpha\beta}$ is a surface tensor so that its material derivative is, in fact, a partial derivative.

$\frac{\partial a}{\partial a_{\alpha\beta}}$ is, in fact, the cofactor of $a_{\alpha\beta}$ given by:

$$\frac{\partial a}{\partial a_{\alpha\beta}} = a^{\alpha\beta} \cdot a \tag{5.2.55}$$

Substitution to Eq. (5.2.54) and rearrangement of the components yields,

$$\frac{Da}{Dt} = 2a\left[V^{\alpha}_{;\alpha} - Wb^{\alpha}_{\alpha}\right] = 2a\ \psi^{\Gamma}_{.\Gamma} \tag{5.2.26}$$

Let us now introduce the rate of deformation tensor, $d_{\Gamma\Lambda}$, by:

$$d_{\Gamma\Lambda} = \frac{1}{2}\left[\psi_{\Gamma\Lambda} + \psi_{\Lambda\gamma}\right] \tag{5.2.57}$$

and the spin tensor $\omega_{\Gamma\Lambda}$ by

$$\omega_{\Gamma\Lambda} = \frac{1}{2}\left[\psi_{\Gamma\Lambda} - \psi_{\Lambda\Gamma}\right] \tag{5.2.58}$$

The rate of deformation and the spin tensors are the symmetric and antisymmetric parts of $\psi_{\Gamma\Lambda}$. The description of the components of $d_{\Gamma\Lambda}$ is simple,

$$d_{\Gamma\Lambda} = \frac{1}{2}(V_{\alpha;\beta} + V_{\beta;\alpha} - 2Wb_{\alpha\beta})\ \delta^{\alpha}_{\Gamma}\ \delta^{\beta}_{\Lambda} \tag{5.2.59}$$

while the description of the components of $\omega_{\Gamma\Lambda}$ from Eq. (5.2.44) is

$$\omega_{\Gamma\Lambda} = \frac{1}{2}(V_{\alpha;\beta} - V_{\beta;\alpha})\ \delta^{\alpha}_{\Gamma}\ \delta^{\beta}_{\Lambda} - \rho_{\alpha}\ \delta^{\alpha}_{\Gamma}\ \delta^{3}_{\Lambda} + \rho_{\beta}\ \delta^{3}_{\Gamma}\ \delta^{\beta}_{\Lambda} =$$

$$= \frac{1}{2}(V_{\alpha;\beta} - V_{\beta;\alpha})\ \delta^{\alpha}_{\Gamma}\ \delta^{\beta}_{\Lambda} - (b_{\alpha\gamma}\ V^{\Gamma} + W_{;\alpha})\ \delta^{\alpha}_{\gamma}\ \delta^{3}_{\Lambda} +$$

$$+ (b_{\beta\gamma}\ V^{\gamma} + W_{;\beta})\ \delta^{3}_{\Gamma}\ \delta^{\beta}_{\Lambda} \tag{5.2.60}$$

Another important element in the description of the deformation of surfaces is the rate of change of the curvature, $b_{\alpha\beta}$. From the definition of $b_{\alpha\beta}$ [Eq. (5.2.6)] we obtain:

$$b_{\alpha\beta} = n_i\ x^{i}_{\alpha;\beta} \tag{5.2.61}$$

The material derivative of Eq. (7.2.61) is then

$$\frac{Db_{\alpha\beta}}{Dt} = \frac{Dn_i}{Dt} x^i_{\alpha;\beta} + n_i \frac{Dx^i_{\alpha;\beta}}{Dt} \tag{5.2.62}$$

From Eqs. (5.2.6) and (5.2.36) it is readily obtained that the first right hand term of Eq. (5.2.62) is zero. In order to calculate the material derivative of $x^i_{\alpha;\beta}$, there is a need to make use of the developed connection between the time material derivatives and the total covariant derivatives (section 2.4). Following the derivation there and noticing that the space is Euclidean results into

$$\frac{Dx^i_{\alpha;\beta}}{Dt} = V^i_{;\alpha\beta} \tag{5.2.63}$$

Substitution into Eq. (5.2.62) of the covariant differentiation of Eq. (5.2.33) and then substitution into Eq. (5.2.13) yields the following expression for $Db_{\alpha\beta}/Dt$,

$$\frac{Db_{\alpha\beta}}{Dt} = n_i V^i_{;\alpha\beta} = b_{\gamma\alpha;\beta}V^\gamma + b_{\gamma\beta} V^\gamma_{;\alpha} + b_{\gamma\alpha} V^\gamma_{;\beta} - b^\gamma_\beta b_{\gamma\alpha} W + W_{;\alpha\beta} \tag{5.2.64}$$

or by using Eqs. (5.2.37) and (5.2.38);

$$\frac{Db_{\alpha\beta}}{Dt} = b^\gamma_\beta \psi_{\gamma\alpha} + \rho_{\alpha;\beta} \tag{5.2.65}$$

Another important variable, in obtaining material derivatives, is the gradient of the normal vector to the surface, $n^i_{,\alpha}$. Once again from section 2.4 we have, for Euclidean space;

$$\frac{Dn^{i}_{,\alpha}}{Dt} = \frac{Dn^{i}}{Dt}_{;\alpha} \tag{5.2.66}$$

From Eqs. (5.2.36) and (5.2.38) we obtain:

$$\frac{Dn^{i}_{;\alpha}}{Dt} = - \left[-\rho^{\gamma}_{;\alpha}\, x^{i}_{\gamma} + b_{\gamma\alpha}\, \dot{\rho}^{\gamma}\, n^{i}\right] \tag{5.2.67}$$

The final expressions in Eqs. (5.2.64) and (5.2.65) can be "translated", as they are, into the τ-system (in opposite e.g. to Eq. (5.2.63)):

$$\frac{Db_{\Gamma\Lambda}}{Dt} = \frac{Db_{\alpha\beta}}{Dt}\, \delta^{\alpha}_{\Gamma}\, \delta^{\beta}_{\Lambda} \tag{5.2.68}$$

5.3 Geometry and Kinematics of Shell

5.3.1 - Geometry of the Shell Structure

A shell is a three-dimensional body with one dimension small, compared to the other two. So, the shell structure can be viewed as some reference surface having material of some thickness on both sides. It is usually assumed that this reference surface is in the middle of the structure and so it is called "the middle surface." Hence, it is possible to describe a position of any material point of the shell by the following system of coordinates:

$$\theta^{\Gamma} = \begin{cases} u^{\alpha} & ; \quad \Gamma = 1, 2 \\ \xi_o & ; \quad \Gamma = 3 \end{cases} \tag{5.3.1}$$

where u^{α} are the material coordinates on the reference surface and and ξ_o is a material coordinate normal to the reference surface and assumed independent of time. It is possible to choose ξ_o as the distance of the material point from the refernce surface at time $t = 0$. Then the distance of any material point from the reference surface at any time will be constrained by:

$$- h(u^{\alpha},t) \leq \xi(\theta^{\Gamma},t) \leq + h(u^{\alpha},t) \tag{5.3.2}$$

where h is half the thickness of shell in that particular point. Most generally, it is possible to describe the position of any material point in the shell by:

$$\tilde{x}(\theta^{\Gamma},t) = x^i(u^{\alpha},t) + y^i(\theta^{\Gamma},t) \tag{5.3.3}$$

where x^i is the position of the middle surface. The additional vector y^i can be decomposed, in the τ-system, into a reference surface part and a part which is normal to the reference surface, or

$$y^i(\theta^\Gamma,t) = x^i_\Gamma\, y^\Gamma = x^i_\alpha\, y^\alpha(\theta^\Gamma,t) + n^i Y(\theta^\Gamma,t) \tag{5.3.4}$$

The component of y^i, which is normal to the reference surface, is the distance of the material point from the reference surface, meaning,

$$\xi(\theta^\Gamma,t) \stackrel{\Delta}{=} y(\theta^\Gamma,t)$$

From Eq. (5.3.3) it is clear that for $\theta^3 = \xi_o = 0$ we obtain:

$$\bar{x}(u^\alpha,0,t) = x^i(u^\alpha,t) \tag{5.3.5a}$$

or, in fact:

$$y^\alpha(u^\alpha,0,t) = 0 \text{ and } Y(u^\alpha,0,t) = 0 \tag{5.3.5b}$$

Without loss of generality, it can be assumed that, at $t = 0$,

$$\bar{x}^i(\theta^\Gamma,0) = x^i(u^\alpha,0) + Y(\theta^\Gamma,0)n^i \tag{5.3.6}$$

This means tht at time $t = 0$, the particular point is on the normal to the surface, so:

$$y^\alpha(\theta^\Gamma,0) = 0 \tag{5.3.7}$$

Let us consider next a few approximating assumptions related to the character of the deformation.

Assumption I: The material points which were on the normal before the deformation will be on the same normal to the surface after deformation. In other words:

$$\text{if: } y^\alpha(\theta^\Gamma,0) = 0 \quad \text{then: } y^\alpha(\theta^\Gamma,t) = 0 \tag{5.3.8}$$

Substitution of this assumption, which is nothing else but the first Kirchhoff-Love assumption (see section 5.1), into Eq. (5.3.3) yields:

$$\bar{x}^i(\theta^\Gamma,t) = x^i(u^\alpha,t) + Y(\theta^\Gamma,t)n^i \tag{5.3.9}$$

From Eq. (5.3.5b) it can be shown that it is possible to expand Y into a power series of ξ_0 in the following manner:

$$Y(\theta^\Gamma,t) = \sum_{n=1}^{\infty} \psi_n(u^\alpha,t) \cdot \xi_0^n = \xi_0 \sum_{n=0}^{\infty} \psi_{n+1}(u^\alpha,t) \cdot \xi_0^n \tag{5.3.10}$$

With regard to Eq. (5.3.10), it is possible to introduce yet another simplifying assumption.

<u>Assumption II</u>: The shell is "sufficiently thin" in order to assume a linear dependence of x^i on ξ_0 then,

$$Y(\theta^\Gamma,t) = \xi_0\ \psi_1(u^\alpha,t) \tag{5.3.11}$$

Substitution of Eq. (5.3.11) into Eq. (5.3.9) and use of the notation $\psi_1 \stackrel{\Delta}{=} \psi$ yield:

$$\bar{x}^i(\theta^\Gamma,t) = x^i(u^\alpha,t) + \xi_0\psi(u^\alpha,t)n^i \tag{5.3.12}$$

It should be remarked that it is possible at this stage to assume a parabolic dependence on ξ_0 and to get a theory which are similar to Reissner's theory. The linear dependence is the most popular in the different approximate shell theories (for this shells).

<u>Assumption III</u>: The change with time of the gradient on the surface of the shell is negligibly small. A different formulation of this assumption can be found in Niordson's work [95]. According to this assumption, points

which are on a surface which is parallel to the middle surface before deformatin will be on a parallel surface after deformation, meaning that ψ is the independent of u^α :

$$\tilde{x}^i(\theta^\Gamma,t) = x^i(u^\alpha,t) + \xi_o\psi(t)n^i \tag{5.3.13}$$

Additional assumptions lead us to the classical formulations.

<u>Assumption IV</u>: The change with time of a distance of a particular point from the middle surface is negligibly small. This is in fact the third Kirchhoff-Love assumption and in our notation it means that ξ (or Y or ψ) are independent of time:

$$\tilde{x}^i(\theta^\Gamma,t) = x^i(u^\alpha,t) + \xi_o\, n^i \tag{5.3.14}$$

Eq. (7.3.14) is the common one (in the literature) for shell theories obeying the Kirchhoff-Love hypotheses.

On the basis of the above four simplifying assumption, several formulations result, for the analysis of thin shells. These formulations are denoted below by capital letters.

<u>Formulation A</u>: This formulation makes use of Assumptino I, only.

<u>Formulation B</u>: This formulation employs Assumption I, and II.

<u>Formulation C</u>: This formulation employs Assumption I, II & III.

<u>Formulation D</u>: This is the classical thin shell theory based on the Kirchhoff-Love hypotheses or Assumptions I, II, III and IV.

<u>Formulation E</u>: There exists a different formulation in the open literature[96], Assumption I (accounts for transverse shear effects). This literature[96, 97] which removes Assumption I (accounts for transverse shear

effects). This formulation starts by assuming that the shell is thin but $y^\alpha \neq 0$ in Eq. (5.3.4). In other words, they assume II without assuming I and obtain from Eqs. (5.3.4) and (5.3.11).

$$\bar{x}^i(\theta^\Gamma,t) = x^i(u^\alpha,t) + \xi_o\bar{y}^i(u^\alpha,t)$$

$$= x^i(u^\alpha,t) + \xi_o[x^i_\alpha\ \bar{y}^\alpha(u^\alpha,t) + \psi(u^\alpha,t)n^i] \quad (5.3.15)$$

Additional Formulation: It is possible also in Eq. (5.3.15) to impose additional assumptions about the shape of ψ, such as Assumptions III and IV. The kinematics of the shell structure can be developed on the basis of any of these assumptions (or others). The corresponding metric for the different formulations are described next, before developing the kinematic expressions.

5.3.2 - The Shell Metric

In this subsection we calculate the transformation tensors and the covarariant components of the metric tensor for every one of the shell formulations which were presented and discussed in the last subsection.

First, for the case of the most general description (without any aproximating assumptions), from Eq. (5.3.3) we obtain the transformation tensor:

$$\bar{x}^i{}_{;\Gamma} = x^i{}_{;\Gamma} + y^i{}_{;\Gamma} \quad (5.3.16)$$

Already in the notation of Eq. (5.3.16) there is a hidden assumption that the system at rest x^i is cartezian. Otherwise, the transformation tensor, which in fact is $\partial x^i/\partial\theta^\Gamma$, would need an additional term which includes Christofel symbols of the space, on the right hand side of Eq. (5.3.16). From Eq. (5.3.4), one can write

$$y^i{}_{;\Gamma} = \{x^i_\gamma\,[y^\gamma{}_{;\alpha} - b^\gamma_\beta\,Y] + n^i[b_{\gamma\alpha}y^\gamma + Y_{;\alpha}]\}\,\delta^\alpha_\Gamma + \{x^i_\gamma\,y^\gamma{}_{;\xi o} + n^i\,Y_{;\xi o}\}\,\delta^3_\Gamma \tag{5.3.17}$$

Similarly to the definitions given by eqs. (5.2.37) and (5.2.38) for $v^i;\Gamma$, we define here the following tensors, for $y^i;\Gamma$

$$\lambda^\alpha_{\cdot\beta} = y^\alpha_{;\beta} - b^\alpha_\beta\,Y \tag{5.3.18}$$

$$\gamma_{\cdot\beta} = b_{\alpha\beta}\,y^\alpha + Y_{;\beta} \tag{5.3.19}$$

By substitution of Eqs. (5.3.17), (5.3.18) and (5.3.19) into Eq. (5.2.16) we obtain

$$\tilde{x}^i{}_{;\Gamma} = \{x^i_\gamma\,[\delta^\gamma_\alpha + \lambda^\gamma_\alpha] + n^i\gamma_\alpha\}\,\delta^\alpha_\Gamma + \{x^i_\gamma\,Y_{;\xi o} + n^i\,Y_{;\xi o}\}\,\delta^3_\Gamma \tag{5.3.20}$$

The covariant metric tensor of the shell is defined by:

$$\bar{g}_{\Gamma\Lambda} = g_{ij}\,\tilde{x}^i{}_{;\Gamma}\,\tilde{x}^i{}_{;\Lambda} \tag{5.3.21}$$

Substitution of Eq. (5.3.20) into Eq. (5.3.21) and rearrangement of the terms yield the following expression:

$$\begin{aligned}\tilde{g}_{\Gamma\Lambda} = {} & \{a_{\alpha\beta} + \lambda_{\alpha\beta} + \lambda_{\beta\alpha} + \lambda^\gamma_\alpha\,\lambda_{\gamma\beta} + \gamma_a\,\gamma_\beta\}\,\delta^\alpha_\Gamma\,\delta^\beta_\Lambda \\ & + \{y_{\alpha;\xi o} + \lambda^\gamma_\alpha\,y_{\gamma;\xi o} + \gamma_\alpha\,Y_{;\xi o}\}\,\delta^\alpha_\Gamma\,\delta^3_\Lambda \\ & + \{y_{\beta;\xi o} + \lambda^\gamma_\beta\,y_{\gamma;\xi o} + \gamma_\beta\,Y_{;\xi o}\}\,\delta^3_\Gamma\,\delta^\beta_\Lambda \\ & + \{y_{\gamma;\xi o}\,y^\gamma_{;\xi o} + Y_{;\xi o}\,Y_{;\xi o}\}\,\delta^3_\Gamma\,\delta^3_\Lambda\end{aligned} \tag{5.3.22}$$

where $a_{\alpha\beta}$ is the metric tensor of the surface, Eq. (5.2.2). The contravariant components of the metric tensor are defined from the following equality:

$$\tilde{g}_{\Gamma\Lambda} \cdot \tilde{g}^{\Lambda\Omega} = \delta_{\Gamma}^{\Omega} \tag{5.3.23}$$

Similar to the surface formulation, it is possible to define here the inverse transformation tensor $\tilde{L}_i^{\Gamma}$ by:

$$\tilde{L}_i^{\Gamma} = g_{ij}\, \tilde{g}^{\Gamma\Lambda}\, \tilde{x}^j{}_{;\Lambda} \tag{5.3.24}$$

So that,

$$\tilde{L}_i^{\Lambda}\, \tilde{x}^i{}_{;\Gamma} = \delta_{\Gamma}^{\Lambda} \tag{5.3.25}$$

and

$$\tilde{g}^{\Lambda\Gamma} = g^{ij}\, \tilde{L}_i^{\Lambda}\, \tilde{L}_j^{\Gamma} \tag{5.3.26}$$

It is worthwhile to notice that the covariant derivative of the shell's metric tensor, Eq. (5.3.22), is not equal to zero. This means that raising and lowering of indices in the space of the shell is not permutable with covariant derivatives. For example for any vector T^{Γ}:

$$T^{\Gamma}_{;\phi} = (g^{\Gamma\Lambda}T_{\Lambda})_{;\phi} \neq g^{\Gamma\Lambda}\, T_{\Lambda;\phi} \tag{5.3.27}$$

<u>Formulation A</u>

We shall formulate now the metric of the shell in view of assumption I (Formulation A). From Eq. (5.3.9) or by direct reduction of Eq. (5.3.20) for $y^{\alpha} = 0$, we obtain for the transformation tensor:

$$\tilde{x}^i{}_{;\Gamma} = \{x^i_{\gamma}[\delta^{\gamma}_{\alpha} - Yb^{\gamma}_{\alpha}] + n^i Y_{;\alpha}\}\, \delta^{\alpha}_{\Gamma} + \{Y_{;\xi o} n^i\}\, \delta^3_{\Gamma} \tag{5.3.28}$$

and for the metric tensor:

$$\bar{g}_{\Gamma\Lambda} = \{a_{\alpha\beta} - 2b_{\alpha\beta}\, Y + Y^2\, b_{\gamma\alpha}\, b^{\gamma}_{\beta} + Y_{;\alpha}\, Y_{;\beta}\}\, \delta^{\alpha}_{\Gamma}\, \delta^{\beta}_{\Lambda}$$

$$+ \{Y_{;\alpha}\, Y_{;\xi o}\}\, \delta^{\alpha}_{\Gamma}\, \delta^{3}_{\Lambda} + \{Y_{;\beta} Y_{;\xi o}\}\, \delta^{3}_{\Gamma}\, \delta^{\beta}_{\Lambda} + \{Y_{;\xi o} Y_{;\xi o}\}\, \delta^{3}_{\Gamma}\, \delta^{3}_{\Lambda}$$

Formulation B

When the second assumptin is imposed (Formulation B), we obtain from Eq. (5.3.12) for the transformation tensor:

$$\bar{x}^{i}_{;\Gamma} = \{x^{i}_{\gamma}[\delta^{\gamma}_{\alpha} - \xi_o\, \psi b^{\gamma}_{\alpha}] + n^{i}\, \xi_o\, \psi_{;\alpha}\}\, \delta^{\alpha}_{\Gamma} + \{\psi\, n^{i}\}\, \delta^{3}_{\Gamma} \qquad (5.3.30)$$

and for the metric tensor:

$$\bar{g}_{\Gamma\Lambda} = \{a_{\alpha\beta} - 2\xi_o\, \psi b_{\alpha\beta} + \xi_o^2\, \psi^2\, b_{\gamma\alpha}\, b^{\gamma}_{\beta} + \xi_o^2\, \psi_{;\alpha}\, \psi_{;\beta}\}\delta^{\alpha}_{\Gamma}\, \delta^{\beta}_{\Lambda}$$

$$+ \{\xi_o\, \psi\, \psi_{;\alpha}\}\, \delta^{\alpha}_{\Gamma}\, \delta^{3}_{\Lambda} + \{\xi o\, \psi\, \psi_{;\beta}\}\, \delta^{3}_{\Gamma}\, \delta^{\beta}_{\Lambda} + \{\psi^2\}\, \delta^{3}_{\Gamma}\, \delta^{3}_{\Lambda} \qquad (5.3.31)$$

Formulation C

When the third assumption is added (Formulation C) we obtain from Eq. (5.3.13) for $x^{i}_{;\Gamma}$:

$$\bar{x}^{i}_{;\Gamma} = \{x^{i}_{\gamma}\, [\delta^{\gamma}_{\alpha} - \xi_o \psi b^{\gamma}_{\alpha}]\}\, \delta^{\alpha}_{\Gamma} + \{\psi\, n^{i}\}\, \delta^{3}_{\Gamma} \qquad (5.3.32)$$

and for the metric tensor:

$$\bar{g}_{\Gamma\Lambda} = \{a_{\alpha\beta} - 2\xi o\, \psi b_{\alpha\beta} + \xi_o^2\, \psi^2 b_{\alpha\beta}\}\, \delta^{\alpha}_{\Gamma}\, \delta^{\beta}_{\Lambda} + \{\psi^2\}\, \delta^{3}_{\Gamma}\, \delta^{3}_{\Lambda} \qquad (5.3.33)$$

Formulation D

For the classical shell approximation theory, we obtain from Eq. (5.3.14):

$$\bar{x}^i_{;\Gamma} = \{x^i_\gamma\,[\delta^\gamma_\alpha - \xi_o b^\gamma_\alpha]\}\ \delta^\alpha_\Gamma + \{n^i\}\ \delta^3_\Gamma \tag{5.3.34}$$

$$\bar{g}_{\Gamma\Lambda} = \{a_{\alpha\beta} - 2\xi_o\ b_{\alpha\beta} + \xi_o^2\ b_{\gamma\alpha}\ b^\gamma_\beta\}\ \delta^\alpha_\Gamma\ \delta^\beta_\Lambda + \delta^3_\Gamma\ \delta^3_\Lambda \tag{5.3.35}$$

Formulation E

Finally, for this formulation, we get slightly different expressions from Eq. (5.3.15). First, for the transformation tensor:

$$\bar{x}^i_{\ ;\Gamma} = \{x^i_\gamma\ [\delta^\gamma_\alpha + \xi o\ \bar{\lambda}^\gamma_\alpha] + n^i \xi o\ [\bar{\gamma}_\alpha]\}\ \delta^\alpha_\Gamma + \{x^i_\gamma\ \bar{y}^\gamma + n^i\psi\}\ \delta^3_\Gamma \tag{5.3.36}$$

and then for the metric tensor:

$$\bar{g}_{\Gamma\Lambda} = \{a_{\alpha\beta} + \xi o\ [\ \bar{\lambda}_{\alpha\beta} + \bar{\lambda}_{\beta\alpha}] + \xi_o^2[\bar{\lambda}^\gamma_\alpha\ \bar{\lambda}_{\gamma\beta} + \bar{\gamma}_\alpha\ \bar{\gamma}_\beta]\}\ \delta^\alpha_\Gamma\ \delta^\beta_\Lambda$$

$$+ \{\bar{y}_\alpha + \xi_o\ [\bar{\lambda}^\gamma_\alpha\ \bar{y}_\gamma + \bar{\gamma}_\alpha\ \psi]\}\ \delta^\alpha_\Gamma\ \delta^3_\Lambda$$

$$+ \{\bar{y}_\beta + \xi o\ [\bar{\lambda}^\gamma_\beta\ \bar{y}_\gamma + \bar{\gamma}_\beta\psi]\}\ \delta^3_\Gamma\ \delta^\beta_\Lambda$$

$$+ \{\bar{y}_\gamma\ \bar{y}^\gamma + \psi^2\}\ \delta^3_\Gamma\ \delta^3_\Lambda \tag{5.3.37}$$

where:

$$\bar{\lambda}^\alpha_{\ .\beta} = (\bar{y}^\alpha_{;\beta} - b^\alpha_\beta\psi) \tag{5.3.38a}$$

$$\bar{\gamma}_\alpha = (b_{\gamma\alpha}\bar{y}^\gamma + \psi_{;\alpha}) \tag{5.3.38b}$$

meaning that $\bar{\lambda}$ and $\bar{\gamma}$ are of the same structure as those defined by Eq. (5.3.18) and (5.3.19), but independent of ξo.

5.3.3 - The Shell Kinematics

The kinematics of shell can be fully described by the components of the rate of the transformation tensor $\psi_{\Gamma\Lambda}$, defined by Eq. (5.2.43), for the τ-system. From Eqs. (5.2.57) and (5.2.58) it is clear that ψ defines exactly the strains and the rotations of the structure. If one is not satisfied with the internal description, there is a need to define the components of the velocity vector at every point of the structure. Similar to the metric analysis of the different possibilities to contract shell theories (see previous subsection), in this subsection the components of the ψ tensor and the components of the velocity vector will be defined for every shell theory formulation of subsection 5.3.1.

Following the kinematic analysis of surfaces, we define now the components of the velocity vector v by:

$$\bar{v}^i = \frac{d\bar{x}^i}{dt} = \frac{D\bar{x}^i}{Dt} \tag{5.3.34}$$

Substitution of the general definitions given by Eqs (5.3.3) and (5.3.4) into Eqs. (5.3.34) yields:

$$\bar{v}^i = \frac{D}{Dt}\,[x^i + x^i_\alpha\, y^\alpha + n^i\, Y] \tag{5.3.35}$$

and through Eqs. (5.2.30), (5.2.31), (5.2.36), (5.3.37) and (5.2.38) one can write

$$\bar{v}^i = v^i + v^i_{,\alpha}\, y^\alpha + x^i_\alpha\, \frac{DY^2}{Dt} - \rho^\gamma x^i_\gamma\, Y + n^i\, \frac{DY}{Dt} \tag{5.3.36}$$

where v^i is the velocity vector of the reference surface. Eq. (5.3.36) can be shortened by making use of the τ-system as follows:

$$\bar{v}^i = v^i + v^i_{;\Gamma}\, y^\Gamma + x^i_\Gamma\, \frac{Dy^\Gamma}{Dt} \tag{5.3.37}$$

We define now the rate of deformation tensor ψ for the general case Eq. (5.3.3). For this case the transformation tensor $\tilde{x}^i_{;\Gamma}$ is defined by Eq. (5.3.20) and the metric tensor by Eq. (5.3.22). From the definition for ψ, Eq. (5.2.43), we obtain

$$\tilde{\psi}_{\Gamma\Lambda} = g_{ij}\, \tilde{x}^i_{;\Gamma}\, \frac{D\tilde{x}^j_{;\Lambda}}{Dt} \tag{5.3.38}$$

Substitution of Eq. (5.3.20) into Eq. (5.3.38) and reordering of terms yield:

$$\begin{aligned}
\psi_{\Gamma\Lambda} = \Big\{ & \psi_{\alpha\beta} + \psi_{\alpha\beta}\lambda^\gamma_\beta + \psi_{\gamma\beta}\lambda^\gamma_\alpha + \psi_{\gamma\delta}\lambda^\gamma_\alpha\lambda^\delta_\beta + a_{\alpha\gamma}\frac{D\lambda^\gamma_\beta}{Dt} + a_{\gamma\delta}\lambda^\gamma_\alpha\frac{D\lambda^\delta_\beta}{Dt} \\
& + \gamma_\alpha \frac{D\gamma_\beta}{Dt} - \rho_\alpha\gamma_\beta + \rho_\beta\gamma_\alpha + \rho_\gamma\left[\gamma_\alpha\lambda^\gamma_\beta - \gamma_\beta\lambda^\gamma_\alpha\right]\Big\}\,\delta^\alpha_\Gamma\,\delta^\beta_\Lambda \\
& + \Big\{\psi_{\gamma\beta}\, y^\gamma_{;\xi_o} + \psi_{\gamma\delta}\lambda^\delta_\beta\, y^\gamma_{;\xi_o} + a_{\gamma\delta}\frac{D\lambda^\delta_\beta}{Dt}\, y^\gamma_{;\xi_o} + \frac{D\gamma_\beta}{Dt}\, Y_{;\xi_o} \\
& + \rho_\beta\lambda^\gamma_\beta\, Y_{;\xi o} - \gamma_\beta y^\gamma_{;\xi o}\big] + \rho_\beta\, Y_{;\xi o}\Big\}\,\delta^3_\Gamma\,\delta^\beta_\Lambda + \Big\{\psi_{\alpha\gamma}\, y^\gamma_{;\xi_o} \\
& + \psi_{\delta\gamma}\lambda^\delta_\beta\, y^\gamma_{;\xi_o} + a_{\alpha\gamma}\frac{Dy^\gamma_{;\xi o}}{Dt} + a_{\gamma\delta}\lambda^\delta_\alpha\frac{Dy^\gamma_{;\xi o}}{Dt} + \gamma_\alpha\frac{DY_{;\xi o}}{Dt} \\
& + \rho_\gamma\left[\lambda^\gamma_\alpha\, Y_{;\xi o} - \gamma_\alpha\, y^\gamma_{;\xi o}\right] - \rho_\alpha\, Y_{;\xi o}\Big\}\,\delta^\alpha_\Gamma\,\delta^3_\Lambda \\
& + \Big\{\psi_{\gamma\delta}\, y^\gamma_{;\xi o}\, y^\delta_{;\xi o} + a_{\gamma\delta}\frac{Dy^\gamma_{;\xi o}}{Dt}\, y^\delta_{;\xi o}\, Y_{;\xi o}\frac{DY_{;\xi o}}{Dt}\Big\}\,\delta^3_\Gamma\,\delta^3_\Gamma
\end{aligned} \tag{5.3.39}$$

where $\psi_{\alpha\beta}$ and ρ_{α} are given by Eqs. (5.2.37) and (5.2.38) and λ^{α}_{β} and γ_{α} are given by Eq. (5.3.38). Note that $a_{\alpha\beta}$ is the metric tensor of the reference surface.

For the sake of completeness we also include the expressions for the rate of deformation, $d_{\Gamma\Lambda}$, and spin, $w_{\Gamma\Lambda}$, tensors according to the definitions of Eqs (5.2.57) and (5.2.58), respectively,

$$
\begin{aligned}
\bar{d}_{\Gamma\Lambda} = \{ & d_{\alpha\beta} + d_{\alpha\delta}\,\lambda^{\delta}_{\beta} + d_{\beta\delta}\,\lambda^{\delta}_{\alpha} + d_{\gamma\delta}\,\lambda^{\gamma}_{\alpha}\,\lambda^{\delta}_{\beta} + \frac{1}{2}\,[a_{\alpha\delta}\,\frac{D\lambda^{\delta}_{\beta}}{Dt} + a_{\delta\beta}\,\frac{D\lambda^{\delta}_{\alpha}}{Dt} \\
& + a_{\gamma\delta}(\lambda^{\gamma}_{\alpha}\,\frac{D\lambda^{\delta}_{\beta}}{Dt} + \lambda^{\delta}_{\beta}\,\frac{D\lambda^{\gamma}_{\alpha}}{Dt}) + \gamma_{\alpha}\,\frac{D\gamma_{\beta}}{Dt} + \gamma_{\beta}\,\frac{D\gamma_{\alpha}}{Dt}]\}\;\delta^{\alpha}_{\Gamma}\,\delta^{\beta}_{\Lambda} \\
& + \{d_{\gamma\beta}\,y^{\gamma}{}_{;\xi o} + d_{\gamma\delta}\,\lambda^{\delta}_{\beta}\,y^{\delta}_{;\xi o} + \frac{1}{2}\,[a_{\beta\gamma}\,\frac{Dy^{\gamma}_{;\xi o}}{Dt} + a_{\gamma\delta}(\frac{D\lambda^{\delta}_{\beta}}{Dt}\,y^{\gamma}_{;\xi o} \\
& + \lambda^{\delta}_{\beta}\,\frac{Dy^{\gamma}_{;\xi o}}{Dt}) + \frac{D\gamma_{\beta}}{Dt}\,Y_{;\xi o} + \gamma_{\beta}\,\frac{DY_{;\xi o}}{Dt}]\}\;(\delta^{3}_{\Gamma}\,\delta^{\beta}_{\Lambda} + \delta^{\beta}_{\Gamma}\,\delta^{3}_{\Lambda}) + \\
& + \{d_{\gamma\delta}\,y^{\gamma}_{;\xi o}\,y^{\delta}_{;\xi o} + a_{\gamma\delta}\,\frac{Dy^{\gamma}_{;\xi o}}{Dt}\,y^{\delta}_{;\xi o} + Y_{;\xi o}\,\frac{DY_{;\xi o}}{Dt}\}\;\delta^{3}_{\Gamma}\,\delta^{3}_{\Lambda} = \\
= \{ & d_{\alpha\beta} + \frac{1}{2}\,[\frac{D\lambda_{\alpha\beta}}{Dt} + \frac{D\lambda_{\beta\alpha}}{Dt} + \frac{D\lambda^{\gamma}_{\alpha}}{Dt}\,\lambda_{\alpha\beta} + \lambda^{\gamma}_{\alpha}\,\frac{D\lambda_{\gamma\beta}}{Dt} + \gamma_{\alpha}\,\frac{D\gamma_{\beta}}{Dt} + \\
& + \gamma_{\beta}\,\frac{D\gamma_{\alpha}}{Dt}\,]\}\;\delta^{\alpha}_{\Gamma}\,\delta^{\beta}_{\Lambda} + \frac{1}{2}\{\frac{Dy_{\beta;\xi o}}{Dt} + \frac{D\lambda^{\gamma}_{\beta}}{Dt}\;\;y_{\gamma;\xi o} + \lambda^{\gamma}_{\beta}\,\frac{Dy_{\gamma;\xi o}}{Dt} + \\
& + \frac{D\gamma_{\beta}}{Dt}\,Y_{;\xi o} + \gamma_{\beta}\,\frac{DY_{;\xi o}}{Dt}\}\;(\delta^{3}_{\Gamma}\,\delta^{\beta}_{\Lambda} + \delta^{\beta}_{\Gamma}\,\delta^{3}_{\Lambda}) + \{\frac{Dy_{\gamma;\xi o}}{Dt}\,y^{\gamma}_{;\xi o} + \\
& + Y_{;\xi o}\,\frac{DY_{;\xi o}}{Dt}\}\;\delta^{3}_{\Gamma}\,\delta^{3}_{\Lambda}
\end{aligned}
\tag{5.3.40}
$$

$$\tilde{w}_{\Gamma\Lambda} = \{w_{\alpha\beta} + w_{\alpha\gamma}\,\lambda^{\gamma}_{\beta} + w_{\gamma\beta}\,\lambda^{\gamma}_{\alpha} + w_{\gamma\delta}\,\lambda^{\gamma}_{\alpha}\,\lambda^{\delta}_{\beta} + \frac{1}{2}\,[a_{\alpha\gamma}\,\frac{D\lambda^{\gamma}_{\beta}}{Dt} - a_{\beta\gamma}\,\frac{D\lambda^{\gamma}_{\alpha}}{Dt} +$$

$$+ a_{\gamma\delta}(\lambda^{\gamma}_{\alpha}\,\frac{D\lambda^{\delta}_{\beta}}{Dt} - \frac{D\lambda^{\gamma}_{\alpha}}{Dt}\,\lambda^{\delta}_{\beta}) + \gamma_{\alpha}\,\frac{D\gamma_{\beta}}{Dt} - \frac{D\gamma_{\alpha}}{Dt}\,\gamma_{\beta}] + \gamma_{\alpha}\,\rho_{\beta} - \gamma_{\beta}\,\rho_{\alpha} +$$

$$+ \rho_{\delta}\,(\gamma_{\alpha}\,\lambda^{\delta}_{\beta} + \gamma_{\beta}\,\lambda^{\delta}_{\alpha})\}\,\delta^{\alpha}_{\Gamma}\,\delta^{\beta}_{\Lambda} + \{w_{\gamma\beta}\,y^{\gamma}_{;\xi o} + w_{\gamma\delta}\,\lambda^{\delta}_{\beta}\,y^{\gamma}_{;\xi o} +$$

$$+ \frac{1}{2}\,[-a_{\beta\gamma}\,\frac{Dy^{\gamma}_{;\xi o}}{Dt} + a_{\gamma\delta}\,(\frac{D\lambda^{\delta}_{\beta}}{Dt}\,y^{\gamma}_{;\xi o} - \lambda^{\delta}_{\beta}\,\frac{Dy^{\gamma}_{;\xi o}}{Dt}) + \frac{D\gamma_{\beta}}{Dt}\,Y_{;\xi o}$$

$$- \gamma_{\beta}\,\frac{DY_{;\xi o}}{Dt}\,] + \rho_{\gamma}(\lambda^{\gamma}_{\beta}\,Y_{;\xi o} - \gamma_{\beta}\,y^{\gamma}_{;\xi o}) + \rho_{\beta}\,Y_{;\xi o}\}(\delta^{3}_{\Gamma}\,\delta^{\beta}_{\Lambda} - \delta^{\beta}_{\Gamma}\,\delta^{3}_{\Lambda})$$

(5.3.41)

In Eqs. (5.3.40) and (5.3.41), $d_{\alpha\beta}$ and $w_{\alpha\beta}$ are the component of the rate of deformation and the spin tensors of the reference surface, defined by Eqs. (5.2.59) and (5.2.60).

It is worth noticing here again that the expressions in Eqs. (5.3.39) - (5.3.41) are exact, but virtually impossible for someone to use them in numerical computations, because of their complexity. However, they may be used to check the accuracy and performance of the other formulations.

Formulation A

We develop now the kinematic expressions according to Assumption I (Formulation A), that is $y^{\gamma} \equiv 0$. The components of the velocity vector are obtained through differentiation of x^{i}, Eq. (5.3.9), or by substitution of the assumption $y^{\gamma} = 0$ into the exact expression of Eq. (5.3.36). Accordingly we obtain:

$$\bar{v}^{i} = v^{i} - \rho^{\gamma}\,x^{i}_{\gamma}\,Y + n^{i}\,\frac{DY}{Dt} \qquad (7.3.42)$$

The rate of transformation tensor is obtained also, by substitution of the right transformation tensor into an Assumption I [from Eq. (5.3.34) into the definition of Eq. (5.3.38), or by reduction of Eq. (5.3.39)]:

$$
\begin{aligned}
\bar{\psi}_{\Gamma\Lambda} = \{ & \psi_{\alpha\beta} - \psi_{\alpha\beta}\, b^{\gamma}_{\beta}\, Y - \psi_{\gamma\beta}\, b^{\gamma}_{\alpha}\, Y + \psi_{\gamma\delta}\, b^{\gamma}_{\alpha}\, b^{\delta}_{\beta}\, Y^2 \\
& - (a_{\alpha\beta} - a_{\delta\gamma}\, b^{\delta}_{\alpha}\, Y)\left(\frac{Db^{\gamma}_{\beta}}{Dt}\, Y + b^{\gamma}_{\beta}\, \frac{DY}{Dt}\right) + Y_{;\alpha}\, \frac{DY_{;\beta}}{Dt} \\
& - \rho_{\alpha}\, Y_{;\beta} + \rho_{\beta}\, Y_{;\alpha} - \rho_{\gamma}\, Y[b^{\gamma}_{\beta}\, Y_{;\alpha} + b^{\gamma}_{\alpha}\, Y_{;\beta}]\}\, \delta^{\alpha}_{\Gamma}\, \delta^{\beta}_{\Lambda} \\
& + \left\{\frac{DY_{;\beta}}{Dt}\, Y_{;\xi o} - \rho_{\gamma}\, b^{\gamma}_{\beta}\, Y\, Y_{;\xi o} + \rho_{\beta}\, Y_{;\xi_o}\right\} \delta^{3}_{\Gamma}\, \delta^{\beta}_{\Lambda} \qquad (5.3.43) \\
& + \left\{Y_{;\alpha}\, \frac{DY_{;\xi o}}{Dt} + \rho_{\gamma}\, b^{\gamma}_{\alpha}\, Y\, Y_{;\xi o} - \rho_{o}\, Y_{;\xi o}\right\} \delta^{\alpha}_{\Gamma}\, \delta^{3}_{\Lambda} + \left\{Y_{;\xi o}\, \frac{DY_{;\xi o}}{Dt}\right\} \delta^{3}_{\Gamma}\, \delta^{3}_{\Lambda}
\end{aligned}
$$

The rate of deformation tensor for the Formulation A is

$$
\begin{aligned}
\bar{d}_{\Gamma\Lambda} = \{ & d_{\alpha\beta} - \frac{Db_{\alpha\beta}}{Dt}\, Y - b_{\alpha\beta}\, \frac{DY}{Dt} + \frac{1}{2}\left(\frac{Db^{\gamma}_{\alpha}}{Dt}\, b_{\gamma\beta}\, Y^2 + \frac{Db_{\gamma\beta}}{Dt}\, b^{\gamma}_{\alpha}\, Y^2\right) \\
& + b^{\gamma}_{\alpha}\, b_{\gamma\beta} + Y\, \frac{DY}{Dt} + \frac{1}{2}\left(Y_{;\alpha}\, \frac{DY_{;\beta}}{Dt} + Y_{;\beta}\, \frac{DY_{;\alpha}}{Dt}\right)\}\, \delta^{\alpha}_{\gamma}\, \delta^{\beta}_{\Lambda} \\
& + \frac{1}{2}\left\{\frac{DY_{;\beta}}{Dt}\, Y_{;\xi o} + Y_{;\beta}\, \frac{DY_{;\xi o}}{Dt}\right\}(\delta^{3}_{\Gamma}\, \delta^{\beta}_{\Lambda} + \delta^{\beta}_{\Gamma}\, \delta^{3}_{\Lambda}) \\
& + \left\{Y_{;\xi o} + \frac{DY_{;\xi o}}{Dt}\right\} \delta^{3}_{\Gamma}\, \delta^{3}_{\Lambda} \qquad (5.3.44)
\end{aligned}
$$

Similarly, the appropriate spin tensor is

$$\bar{w}_{\gamma\Lambda} + \{w_{\alpha\beta} + (\psi_{\gamma\alpha}\, b^{\gamma}_{\beta} - \psi_{\gamma\beta}\, b^{\gamma}_{\alpha})\, Y + [w_{\gamma\delta}\, b^{\gamma}_{\alpha}\, b^{\delta}_{\beta} + \tfrac{1}{2}(b_{\delta\alpha} \frac{Db^{\delta}_{\beta}}{Dt}$$

$$- b_{\delta\beta} \frac{Db^{\delta}_{\alpha}}{Dt})]\, Y^2 + \frac{1}{2}\, [Y_{;\alpha} \frac{DY_{;\beta}}{Dt} - Y_{;\beta} \frac{DY_{;\alpha}}{Dt}]$$

$$+ \rho_{\beta}\, Y_{;\alpha} - \rho_{\alpha}\, Y_{;\beta} - \rho_{\gamma}(b^{\delta}_{\beta}\, Y\, Y_{;\alpha} - b^{\delta}_{\alpha}\, Y\, Y_{;\beta})\}\, \delta^{\alpha}_{\Gamma}\, \delta^{\beta}_{\Lambda}$$

$$+ \{\frac{1}{2}(\frac{DY_{;\beta}}{Dt}\, Y_{;\xi o} - Y_{;\beta} \frac{DY_{;\xi o}}{Dt}) - \rho_{\gamma}\, b^{\gamma}_{\beta}\, Y\, Y_{;\xi o}$$

$$+ \rho_{\beta}\, Y_{;\xi o}\}\, (\delta^{3}_{\Gamma}\, \delta^{\beta}_{\Lambda} - \delta^{\beta}_{\Gamma}\, \delta^{3}_{\Lambda}) \tag{5.3.45}$$

Study of Eq. (5.3.44) reveals that, in view of assumption I, there exist shear strains through the thickness. These strains are equal to zero in the reference surface [from Eq. (5.3.5b)], but different from zero anywhere else and their value depends on the distance from the reference surface. It is interesting to notice also that these components depends only on the gradients of Y and on the time rate of change of these gradients.

Formulation B

If we assume now a linear dependence of Y on ξo, meaning that we impose the second assumption also, we obtain the components of the velocity vector through differentiation of Eq. (5.3.12), or by substitution of Eq. (5.3.11) into Eq. (5.3.42), for this case

$$\bar{v}^{i} = v^{i} - \xi_{o}\, \psi\, \rho^{\gamma}\, x^{i}_{\gamma} + \xi_{o} \frac{D\psi}{Dt}\, n^{i} \tag{5.3.46}$$

One should not confuse ψ in Eq. (5.3.46), which is a scalar, with the components of the rate of the transformation tensor, $\bar{\psi}_{\Gamma\Lambda}$.

In view of assumption II, the components of the rate of transformation tensor, $\bar{\psi}_{\Gamma\Lambda}$, are obtain from substitutin of Eq. (5.3.11) into Eq. (5.3.43), or

$$\bar{\psi}_{\Gamma\Lambda} = \{\psi_{\alpha\beta} - \xi_o[\psi(\psi_{\alpha\gamma}\ b^{\gamma}_{\beta} + \psi_{\gamma\beta}\ b^{\gamma}_{\alpha} + a_{\alpha\gamma}\frac{Db^{\gamma}_{\beta}}{Dt}) + b_{\alpha\beta}\frac{D\psi}{Dt}$$

$$+ \rho_{\beta}\ \psi_{;\alpha} - \rho_{\alpha}\ \psi_{;\beta}] + \xi_o^2[\psi^2(\psi_{\gamma\delta}\ b^{\gamma}_{\alpha}\ b^{\delta}_{\beta} + b_{\gamma\alpha}\frac{Db^{\gamma}_{\beta}}{Dt})$$

$$+ b_{\gamma\alpha}\ b^{\gamma}_{\beta}\frac{D\psi}{Dt} + \psi_{;\alpha}\frac{D\psi_{;\beta}}{Dt} - \rho_{\gamma}\ \psi(b^{\gamma}_{\beta}\ \psi_{;\alpha} + b^{\gamma}_{\alpha}\ \psi_{;\beta})]\}\ \delta^{\alpha}_{\Gamma}\ \delta^{\beta}_{\Lambda}$$

$$+ \{\psi\rho_{\beta} - \xi_o[\psi^2 b^{\gamma}_{\beta}\ \rho_{\gamma} - \frac{D\psi_{;\beta}}{Dt}\cdot\psi]\}\ \delta^3_{\Gamma}\ \delta^{\beta}_{\Lambda}$$

$$+ \{-\psi\rho_{\alpha} + \xi_o\ [\psi^2\ b^{\gamma}_{\alpha}\ \rho_{\gamma} + \psi_{;\alpha}\frac{D\psi}{Dt}]\}\ \delta^{\alpha}_{\Gamma}\ \delta^3_{\Lambda} + \{\psi\cdot\frac{D\psi}{Dt}\}\ \delta^3_{\Gamma}\ \delta^3_{\Lambda} \qquad (5.3.47)$$

Similarly, the rate of deformation tensor, $\bar{d}_{\Gamma\Lambda}$, for assumption II, is obtain by substitution of Eq. (5.3.11) into Eq. (5.3.44), or

$$\bar{d}_{\Gamma\Lambda} = \{d_{\alpha\beta} - \xi_o\frac{Db_{\alpha\beta}}{Dt} + b_{\alpha\beta}\frac{D\psi}{Dt}] + \xi_o^2[\frac{1}{2}\ \psi^2\ \frac{Db^{\gamma}_{\alpha}}{Dt}\ b_{\gamma\beta} + \frac{Db_{\gamma\beta}}{Dt}\ b^{\gamma}_{\alpha})$$

$$+ b^{\gamma}_{\alpha}\ b_{\gamma\beta}\ \psi\frac{D\psi}{Dt} + \frac{1}{2}(\psi_{;\alpha}\ \frac{D\psi_{;\beta}}{Dt} + \psi_{;\beta}\ \frac{D\psi_{;\alpha}}{Dt}\ )]\}\ \delta^{\alpha}_{\Gamma}\ \delta^{\beta}_{\Lambda}$$

$$+ \{\frac{1}{2}\ \xi_o\ (\frac{D\psi_{;\beta}}{Dt}\ \psi + \psi_{;\beta}\ \frac{D\psi}{Dt})\}\ (\delta^3_{\Gamma}\ \delta^{\beta}_{\Lambda} + \delta^{\beta}_{\Gamma}\ \delta^3_{\Lambda}) + \{\psi\ \frac{D\psi}{Dt}\ \}\ \delta^3_{\Gamma}\ \delta^3_{\Lambda} \qquad (5.3.48)$$

The components of the spin tensor $\bar{\omega}_{\Gamma\Lambda}$, for this case, are obtained from subsitution of Eq. (5.3.11) into Eq. (5.3.45), or

$$\tilde{w}_{\Gamma\Lambda} = \{w_{\alpha\beta} + \xi_o[\psi(\psi_{\gamma\alpha}\, b^{\gamma}_{\beta} - \psi_{\gamma\beta}\, b^{\gamma}_{\alpha}) + \rho_\beta\, \psi_{;\alpha} - \rho_\alpha\, \psi_{;\beta}]$$

$$+ \xi^2_o[\psi^2(w_{\gamma\delta}\ \beta^{\gamma}_{\delta}\ b^{\gamma}_{\alpha}\ b^{\delta}_{\beta} + \tfrac{1}{2}(b_{\delta\alpha}\frac{Db^{\delta}_{\beta}}{Dt} - b_{\delta\beta}\frac{Db^{\delta}_{\alpha}}{Dt})) + \tfrac{1}{2}(\psi_{;\alpha}\frac{D\psi_{;\beta}}{Dt}$$

$$(5.3.49)$$

$$- \psi_{;\beta}\frac{D\psi_{;\alpha}}{Dt}) - \rho_\delta(b^{\delta}_{\beta}\ \psi\ \psi_{;\alpha} - b^{\delta}_{\alpha}\ \psi\ \psi_{;\beta})]\}\ \delta^{\alpha}_{\Gamma}\ \delta^{\beta}_{\Lambda}$$

$$+ \{\psi\rho_\beta + \xi_o\ [\tfrac{1}{2}(\frac{D\psi_{;\beta}}{Dt} \cdot \psi - \psi_{;\beta}\frac{D\psi}{Dt}) - \psi^2\ \rho_\gamma\ b^{\gamma}_{\beta}]\}\ (\delta^3_\Gamma\ \delta^\beta_\Lambda - \delta^\beta_\Gamma\ \delta^3_\Lambda)$$

In both Eqs. (5.3.48) and (5.3.47) it is customary to neglect the terms which are multiplied by ξ_o^2, in accordance with the assumption that only the terms which are linear in ξ_o were kept before. Similar to the results under assumption I (Formulation A), here also the shear strains, $\tilde{d}_{3\Gamma}$, vanish on the reference surface. This strain components are called by Reissner [5.17] "the surface shear couples. The appearances of these shear couples causes surfaces which are parallel to the reference surface before the deformation not to remain parallel to it after deformation. This is so because surface gradient of ψ does not vanish. It is customary to give a physical significance to the function ψ [5.58] through the definition:

$$\psi = \frac{h(u^\alpha,\ t)}{h(u^\alpha, 0)} \qquad (5.3.50)$$

Clearly then, the term $\xi_o\ \psi$ is the distance of a particular point from the reference surface, at time t. It is clear now, that the nonvanishing of the gradient of ψ, ($\psi_{,\alpha}$) means that the change of thickness of the shell is inhomogeneous (nonconstant).

Formulation C

If we superimpose now assumption III, that is, the independence of ψ of u^α , then (as explained before) the surfaces which are parallel to the reference surface will remain parallel after deformation. This means that the change of thickness all over the shell will be the same. This does not limit the shell to be of the same thickness, throughout. According to assumption III, the corresponding velocity vector is obtained by differentiation of Eqs. (5.3.13), or

$$\bar{v}^i = v^i - \xi_o \psi \rho^\gamma x^i_\gamma + \xi_o \frac{D\psi}{Dt} n^i \tag{5.3.51}$$

This Eq. (7.3.51) is of course the same expression as Eq. (5.3.46) for Formulation B. In contrast though, $\bar{\psi}_{\Gamma\Lambda}$, which is obtained by a reduction of Eq. (5.3.47), has a much simpler expression,

$$\begin{aligned}\bar{\psi}_{\Gamma\Lambda} = \{\psi_{\alpha\beta} - \xi_o \, [\psi(\psi_{\alpha\gamma} \, b^\gamma_\beta + \psi_{\gamma\beta} \, b^\gamma_\alpha + a_{\alpha\gamma} \frac{Db^\gamma_\beta}{Dt}) + b_{\alpha\beta} \frac{D\psi}{Dt} \,] \\ + \xi_o^2 [\psi^2(\psi_{\gamma\delta} \, b^\gamma_\alpha \, b^\delta_\beta + b_{\gamma\alpha} \frac{Db^\gamma_\beta}{Dt}) + b_{\gamma\alpha} \, b^\gamma_\beta \, \psi \frac{D\psi}{Dt}]\} \, \delta^\alpha_\Gamma \, \delta^\beta_\Lambda \\ + \{\psi \, \rho_\beta - \xi_o \, [\psi^2 \, b^\gamma_\beta \, \rho_\gamma]\} \, (\delta^3_\Gamma \, \delta^\beta_\Lambda - \delta^\beta_\Gamma \, \delta^3_\Lambda) + \{\psi \frac{D\psi}{Dt}\} \, \delta^3_\Gamma \, \delta^3_\Lambda\end{aligned} \tag{5.3.52}$$

Correspondingly, the components of the rate of the deformation tensor $\bar{d}_{\Gamma\Lambda}$, for this formulation are [from Eq. (5.3.48)]

$$\begin{aligned}\bar{d}_{\Gamma\Lambda} = \{d_{\alpha\beta} - \xi_o[\psi \frac{Db_{\alpha\beta}}{Dt} + b_{\alpha\beta} \frac{D\psi}{Dt}] + \xi_o^2 \, [\frac{1}{2} \, \psi^2(\frac{Db^\gamma_\alpha}{Dt} \, b_{\gamma\beta} + \frac{Db_{\gamma\beta}}{Dt} \, b^\gamma_\alpha) \\ + b^\gamma_\alpha \, b_{\gamma\beta} \, \psi \frac{D\psi}{Dt}]\} \, \delta^\alpha_\Gamma \, \delta^\beta_\Lambda + \{\psi \frac{D\psi}{Dt}\} \, \delta^3_\Gamma \, \delta^3_\Lambda\end{aligned} \tag{5.3.53}$$

Similarly, the components of the spin tensor $\tilde{w}_{\Gamma\Lambda}$, are obtained from Eq. (5.3.49):

$$\tilde{w}_{\Gamma\Lambda} = \{w_{\alpha\beta} + \xi_o[\psi(\psi_{\gamma\alpha}\, b^{\gamma}_{\beta} - \psi_{\gamma\beta}\, b^{\gamma}_{\alpha})] + \xi_o^2[\psi^2(\psi_{\gamma\delta}\, b^{\gamma}_{\alpha}\, b^{\delta}_{\beta} +$$

$$+ \frac{1}{2}(b_{\delta\alpha} \frac{Db^{\delta}_{\beta}}{Dt} - b_{\delta\beta} \frac{Db^{\delta}_{\alpha}}{Dt})]]\}\ \delta^{\alpha}_{\Gamma}\, \delta^{\beta}_{\Lambda}$$

$$+ \{\psi\, \rho_{\beta} - \xi_o[\psi^2\, \rho_{\gamma}\, b^{\gamma}_{\beta}]\}\ (\delta^3_{\Gamma}\, \delta^{\beta}_{\Lambda} - \delta^{\beta}_{\Gamma}\, \delta^3_{\Lambda}) \qquad (5.3.54)$$

As expected, assumption III causes the surface shear couples to vanish. A formulation (development) based on Eqs. (5.3.52) - (5.3.54), with the term multiplyng ξ^2 neglected, can be found in Ref. 98.

Formulation D

The addition of Assumption IV bring us to the rate formulation of the classical shell theory, which includes all the Kirchhoff-Love hypotheses. Imposition of assumption IV, neglectng the change of thickness of shell with time ($\psi = 1$), into Eqs. (5.3.51) - (5.3.54) yields the following expressions:

for the velocity vector

$$\tilde{v}^i = v^i - \xi_o\, \rho^{\gamma}\, x^i_{\gamma} \qquad (5.3.55)$$

for the rate of transformation tensor:

$$\tilde{\psi}_{\Gamma\Lambda} = \{\psi_{\alpha\beta} - \xi_o[(\psi_{\alpha\gamma}\, b^{\gamma}_{\beta} + \psi_{\gamma\beta}\, b^{\gamma}_{\alpha} + a_{\alpha\gamma} \frac{Db^{\gamma}_{\beta}}{Dt})] +$$

$$+ \xi_o^2[(\psi_{\gamma\delta}\, b^{\gamma}_{\alpha}\, b^{\delta}_{\beta} + b_{\gamma\alpha} \frac{Db^{\gamma}_{\beta}}{Dt})\}\ \delta^{\alpha}_{\Gamma}\, \delta^{\beta}_{\Lambda}$$

$$+ \{\rho_{\beta} - \xi_o[\rho_{\gamma}\, b^{\gamma}_{\beta}]\}\ (\delta^3_{\Gamma}\, \delta^{\beta}_{\Lambda} - \delta^{\beta}_{\Gamma}\, \delta^3_{\Lambda}) \qquad (5.3.56)$$

and for the rate of deformation tensor:

$$\tilde{d}_{\Gamma\Lambda} = \{d_{\alpha\beta} - \xi_o[\frac{Db_{\alpha\beta}}{Dt}] + \xi_o^2[\frac{1}{2}(\frac{Db_{\alpha}^{\gamma}}{Dt} b_{\gamma\beta} + \frac{Db_{\gamma\beta}}{Dt} b_{\alpha}^{\gamma})]\ \delta_{\Gamma}^{\alpha}\ \delta_{\Lambda}^{\beta} \tag{5.3.57}$$

the components of the spin tensor remain unchanged, and they are given by Eq. (5.3.54).

A formulation with displacement vectors as given by Eqs. (5.3.55) - (5.3.57), and with neglecting the terms which are multiplied by ξ_0^2, is the classical formulation and appear in many works (see subsection 5.1).

Formulation E

Let us formulate next the kinematics for Formulation E, for which the only requirement, from a physical point of view, is that the shell be thin and from a mathematical point that the expressions have a linear dependence on ξ_0. The components of the velocity vector are obtained through time differentiation of Eq. (5.3.15).

$$\tilde{v}^i = v^i + \xi_o(v^i_{;\alpha}\ \tilde{y}^{\alpha} + x^i_{\alpha} \frac{D\tilde{y}^{\alpha}}{Dt} - \rho^{\gamma}\ x^i_{\gamma}\ \psi + \frac{D\psi}{Dt}\ n^i) \tag{5.3.58}$$

The components of the rate of transformation, rate of deformation and spin tensors are obtained by an appropriate reduction of Eqs. (5.3.39) - (5.3.41), that is in every place instead of λ^{α}_{β} we substitute $\xi_o\ \tilde{\lambda}^{\alpha}_{\beta}$ and instead of γ_{α} we substitute $\xi_0\ \tilde{\gamma}_{\alpha}$, where $\tilde{\lambda}^{\alpha}_{\beta}$ and $\tilde{\gamma}_{\alpha}$ are defined in Eq. (5.3.38). A kinematic development, based on this formulation and expressed in finite displacements and rotations, can be bound in Ref. 96

5.3.4 - Shift to the τ-System

The coordinate systems of the shell space $\bar{x}^i$, which were introduced in subsection 5.3.1, are not well adjusted for the introduction and development of the physical principles to follow. The reason for this is that, on one hand the metric is confusing and of special character (like the nonvanishing of covariant derivative of the metric tensor) and on the other hand it is impossible to compare quantities which are defined in different coordinate systems. The solution to these difficulties is accomplished by shifting the different kinematic parameters to one coordinate system which is of a simple metric and obeys the Ricci Lema. Such a coordinate system is the τ-system which was defined in subsection 5.2.1.

The component of any space vector A^i can be expressed in both systems, the τ-system or the different $\tilde{x}^i$ systems, according to

$$A^i = x^i_\Gamma \, a^\Gamma = \tilde{x}^i{}_{;\Lambda} \tilde{a}^\Lambda \tag{5.3.59}$$

From this, we can obtain the connection between the $\underline{a}$ - components and the $\underline{\tilde{a}}$ - components, or

$$a^\Gamma = L^\Gamma_i \, \tilde{x}^i{}_{;\Lambda} \tilde{a}^\Gamma \tag{5.3.60}$$

Following Ref. [98], we introduce the notation:

$$\mu^\Gamma_\Lambda \overset{\Delta}{=} L^\Gamma_i \, \tilde{x}^i{}_{;\Lambda} \tag{5.3.61a}$$

$$\mu^{-1\Gamma}{}_\Lambda \overset{\Delta}{=} \tilde{L}^\Gamma_i \, x^i_\Lambda \tag{5.3.61b}$$

From their definition, Eq. (7.3.61), tensors μ and μ^{-1} are nothing else but euclidean shifters [99], and they will be used here to transfer the tensor components from the different shell systems to subsection 5.3.1 to the τ-system. The shifters μ and μ^{-1} are absolute double tensors, and they obey the orthogonality conditions,

$$\mu^\Gamma_\Lambda \cdot \mu^{-1\Lambda}{}_\Omega = \delta^\Gamma_\Omega \quad ; \quad \mu^\Gamma_\Lambda \cdot \mu^{-1\Omega}{}_\Gamma = \delta^\Omega_\Lambda \tag{5.3.62}$$

Also from Ref. [99], the metric tensor is transferred from one system to the other by:

$$\tilde{g}_{\Gamma\Lambda} = \mu^{\pi}_{\Gamma}\, \mu^{\phi}_{\Lambda}\, a_{\pi\phi} \tag{5.3.63}$$

in other words, the metric is conserved during the transformation.

As mentioned earlier, the objective is to transfer the kinematic expressions of the last subsection to the τ-system. A short study of subsection 5.3.3 reveals that the shift of the rate of transformation tensor, $\tilde{\psi}$, suffices to give all the needed internal information about the character of the deformation. The definition of $\tilde{\psi}$, from Eq. (5.2.43), is:

$$\tilde{\psi}_{\Gamma\Lambda} = g_{ij}\, \tilde{x}^{i}_{;\Gamma}\, \frac{D\tilde{x}^{i}_{;\Lambda}}{Dt} \tag{5.3.64}$$

It is clear therefore that, in order to transfer $\tilde{\psi}$ it is necessary to shift the transformation $\tilde{x}^{i}_{;\Gamma}$ and their material derivatives. We further note that we are not discussing a "legal" transformation from system to system, therefore it is impossible to perform a direct transformation of tensor $\tilde{\psi}$ without taking into account its components.

From the definitions of x^{i}_{Γ} and L^{Γ}_{i} in the τ-system, Eqs. (5.2.23) - (5.2.24), and from Eq. (5.3.61) it is clear that the shift of the transformation tensor is taken place by:

$$\tilde{x}^{i}_{;\Gamma} = x^{i}_{\Lambda}\, \mu^{\Lambda}_{\Gamma} \tag{5.3.65}$$

For the material derivtive of $\tilde{x}^{i}_{;\Lambda}$, which is, in fact, the covariant derivative of the velocity vector in the shell system, one must take a derivative of Eq. (5.3.65) as is,

$$\frac{D\tilde{x}^{i}_{;\Gamma}}{Dt} = \frac{D(x^{i}_{\Lambda}\, \mu^{\Lambda}_{\Gamma})}{Dt} \tag{5.3.66}$$

From the chain rule and Eq. (5.2.47), one obtains

$$\frac{D\tilde{x}^i_{;\Gamma}}{Dt} = v^i_{;\Lambda}\ \mu^\Lambda_\Gamma + x^i_\Lambda \frac{D\mu^\Lambda_\Gamma}{Dt} \tag{5.3.67}$$

Substitution of Eqs. (5.3.65) and (5.3.67) into Eq. (5.3.64) yields:

$$\tilde{\psi}_{\Gamma\Lambda} = g_{ij}\ x^i_\pi\ \mu^\pi_\Gamma\ [v^j_{;\Omega}\ \mu^\Omega_\Gamma + x^i_\Omega \frac{D\mu^\Omega_\Lambda}{Dt}] \tag{5.3.68}$$

Finally, use of Eqs. (5.2.43) and (5.2.25) results into the form

$$\tilde{\psi}_{\Gamma\Lambda} = \mu_\Gamma\ \mu_\Lambda\ \psi_{\pi\Omega} + a_{\pi\Omega}\ \mu_\Gamma \frac{D\mu^\Omega_\Lambda}{Dt} \tag{5.3.69}$$

The mixed components of $\tilde{\psi}$ are defined by:

$$\tilde{\psi}^{\phi}_{\cdot\Lambda} \overset{\Delta}{=} \tilde{g}^{\phi\Gamma}\ \tilde{\psi}_{\Gamma\Lambda}\ , \tag{5.3.70}$$

and from the inverse operation, in Eq. (5.3.63), we have

$$\tilde{\psi}^{\phi}_{\cdot\Lambda} = a^{\lambda\Sigma}\ \mu^{-1\phi}_{\ \ \Lambda}\ \mu^{-1\Gamma}_{\ \ \Sigma}\ \tilde{\psi}_{\Gamma\Lambda} \tag{5.3.71}$$

Substitution of Eq. (5.3.69) into Eq. (5.3.71) and reordering of the terms with the help of Eq. (5.3.62) yield the following expression:

$$\tilde{\psi}^{\phi}_{\cdot\Lambda} = \mu^\Omega_\Lambda\ \mu^{-1\phi}_{\ \ \lambda}\ \psi^\lambda_{\cdot\Omega} + \mu^{-1\phi}_{\ \ \Omega} \frac{D\mu^\Omega_\Lambda}{Dt} \tag{5.3.72}$$

With the help of Eqs. (5.3.69) and (5.3.72) it is possible to express all the kinematic parameters in the τ-system. The shape of the tensors μ and its inverse, μ^{-1}, will depend on the definition of the shell system, or, in fact, on the definition of $\tilde{x}^i$, according to the different assumptions in subsection 5.3.1.

At this point, one may ask if there is a need for all the manipulations of the kinematics in the preceding subsections, since it suffices to find the expressions for μ and μ^{-1} and by these to uniquely define all the kinematics. However, inspite of the simplicity of the expressions in Eq. (5.3.61), it is not simple at all to find the expressions for the shift tensors, especially for μ^{-1}.

For the components of μ we obtain a closed form expression by substitution of the general definition of $\tilde{x}^{i}_{;\Gamma}$ from Eq.. (5.3.20) into Eq. (5.3.61a), or

$$\mu^{\Gamma}_{\Lambda} = \{\delta^{\Gamma}_{\alpha} + \lambda^{\Gamma}_{.\alpha}\}\ \delta^{\alpha}_{\Lambda} + y^{\Gamma}_{;\xi o}\ \delta^{3}_{\Lambda} \tag{5.3.73}$$

where,

$$\gamma_{\alpha} \overset{\Delta}{=} \lambda^{3}_{.\alpha} \quad ; \quad y^{3} \overset{\Delta}{=} y \tag{5.3.74}$$

The expression in Eq. (5.3.73) is exact, and it is possible to introduce it into any one of the five formulations (A-E) or into additional formulations. As an example, we may substitute the classical assumptions (I-IV) into Eq. (5.3.73). In this case, $y^{i} = \xi_{o}\ n^{i}$ and

$$\lambda^{\Gamma}_{.\alpha} = -b^{\gamma}_{\alpha}\ \xi_{o}\ \delta^{\Gamma}_{\alpha} \quad ; \quad y^{\gamma}_{;\xi o} = \delta^{\Gamma}_{3} \tag{5.3.75}$$

Substitution of Eq. (5.3.75) into Eq. (5.3.73) for μ:

$$\mu^{\Gamma}_{\Lambda} = \{\delta^{\gamma}_{\alpha} - \xi_{o}\ b^{\gamma}_{\alpha}\}\ \delta^{\alpha}_{\Lambda}\ \delta^{\Gamma}_{\gamma} + \delta^{3}_{\Lambda}\ \delta^{\Gamma}_{3} \tag{5.3.76}$$

which is the same expression as in Refs. [98] and [99].

For the inverse shift tensor, μ^{-1}, it is possible to make a similar development as for μ, that is substitution of Eq. (5.3.20) into the definition of $\bar{L}_i^{\Gamma}$, Eq. (5.3.24), and a repeat substitution into Eq. (5.3.61b) yields,

$$\mu^{-1}{}^{\Lambda}_{\Gamma} = \bar{g}^{\Lambda\phi}\{(a_{\alpha\beta}\delta^{\beta}_{\gamma} + \lambda_{\Gamma\alpha})\ \delta^{\alpha}_{\phi} + y_{\Gamma;\xi o}\ \delta^{3}_{\phi}\} \tag{5.3.77}$$

In contrast to the expression for μ, in Eq. (5.3.73), Eq. (5.3.77) includes the metric tensor of the shell and is not expressible in terms of parameters for τ-system, only. In order to overcome this problem for the full Kirchhof-Love assumptions (Formulation E), Refs. [98] and [99] developed μ^{-1} in terms of power series in ξo, and kept only the linear term. If we assume linearity in ξo, and kept meaning neglecting the term multiplied by ξ_o^2 in Eq. (5.3.35), and substitute Eq. (5.3.75) into Eq. (5.3.77), we obtain the same expression for the approximate μ^{-1} as in Refs. [98] and [99]. This expression is

$$\mu^{-1}{}^{\Lambda}_{\Gamma} = \{\delta^{\gamma}_{\beta} + \xi o\ b^{\gamma}_{\beta}\}\ \delta^{\beta}_{\Gamma}\ \delta^{\Lambda}_{\gamma} + \delta^{3}_{\Gamma}\ \delta^{\Lambda}_{3} \tag{5.3.78}$$

At this state of our development, we shall not make any assumptions or approximations for the expression of μ^{-1}, and we shall keep the full expansion, as given by Eq. (5.3.77).

5.4 - The Principle of the Rate of Virtual Power

5.4.1 - The Principle of Virtual Power and the Equation of Equilibrium

The principle of virtual power (virtual velocities), as applied to shells, may be stated, in a similar manner as in the three dimensional formulations, or

$$\int_V \tilde{\sigma}^{\Gamma\Lambda}\, \delta\, \tilde{\psi}_{\Gamma\Lambda}\, dV - \int_V \rho\, \tilde{B}^{\Gamma}\, \delta\tilde{v}_{\Gamma}\, dV - \int_{A_1} \gamma\, \tilde{F}^{\Gamma}\, \delta\tilde{v}_{\Gamma}\, dA_1 = 0 \qquad (5.4.1)$$

where all the parameters are referred to the material system of the shell space

$\tilde{\sigma}^{\Gamma\Lambda}$ - are the components of the Cauchy stress tensor

$\tilde{\psi}_{\Gamma\Lambda}$ - are the components of the rate of transformation tensor

$\tilde{B}^{\Gamma}$ - are the components of body forces per unit mass

$\tilde{F}^{\Gamma}$ - are the components of the external forces on the boundaries per unit area mass.

$\tilde{v}^{\Gamma}$ - are the components of the velocity vector in the shell system.

ρ - volume mass density

γ - mass density per unit area

In order to simplify the first discussion, we limit ourselves to the case in which the external forces are applied only on the shell bounding surfaces ($\xi o = \pm h$). Moreover, we assume that the shell is supported (constrained against translation) around the boundary of the reference surface in the transverse direction. A volume is defined in the shell system, Eq. (5.3.1) by

$$dV = \sqrt{\tilde{g}}\, d\theta^1\, d\theta^2\, d\theta^3 \overset{\Delta}{=} \tilde{g}\, d\theta^1\, d\theta^2\, d\xi_o \qquad (5.4.2)$$

where $\tilde{g}$ is the metric determinant of the shell $[\sqrt{g} = \det(g_{\Gamma\Lambda})]$. Eq. (5.4.2) can be written also in the following way:

$$dV = d\tilde{a}\, d\xi_o \qquad (5.4.3)$$

where $d\tilde{a}$ is a differential area at a distance ξ_o from the reference surface.

Without losing generality, we can write for $\sqrt{\tilde{g}}$:

$$\sqrt{\tilde{g}} = \sqrt{g} \cdot Z(\xi_o) \qquad (5.4.4)$$

where g is the metric determinant in the τ-system, and Z is a function of ζ_o, whose shape changes according to the definition of particular (τ) system, or actually according to the different assumptions about the character of the deformation. In view of Eqs. (5.4.1) and (5.4.2), we can state for dV:

$$dV = \sqrt{g} \cdot Z(\xi_o)\, d\theta^1\, d\theta^2\, d\xi_o = Z(\xi_o)\, da\, d\xi_o \qquad (5.4.5)$$

where da is a differential area on the references surface. In a similar manner we obtain the following expression for dA (differential area on the bounding surface)

$$dA_i = Z(\pm h)\, da \qquad (5.4.6)$$

The expressions appearing as integrands in Eq. (5.4.1) need additional treatment also. It will be convenient to transfer all tensor quantities from the shell system to the τ-system. For the first integral in Eq.

(5.4.1), by making use of Eqs. (5.3.61), (5.3.69), (5.3.62), by reordering the terms and by substituting Eq. (5.4.5), we obtain the following

$$\int_V \tilde{\sigma}^{\Gamma\Lambda} \, \delta\tilde{\psi}_{\Gamma\Lambda} \, dV = \int_V \sigma^{\pi\phi}[\delta\psi_{\pi\phi} + a_{\pi\Omega}(\nu^{-1})^{\Lambda}_{\phi} \, \delta\frac{D\nu^{\Omega}_{\Lambda}}{Dt}] \, Z(\xi_o) \, d\xi_o da \qquad (5.4.7)$$

At this stage, we introduce the first approximatin by neglecting the terms multiplied by ξ_0^2 [inside the brackets of the Eq. (5.4.7)]. This linearization does not affect the term $\delta\psi_{\rho\phi}$, which is independent of ξ_o. The approximation only affects the second term in the brackets, a term which will always contain certain higher powers of ξ_o (but at least ξ_o^2). In view of this approximation, the term in the square brackets is at most linear in ξ_o. Hence, we can denote the coefficients of ξ_o by $\delta\kappa_{\pi\phi}$ and the one which is free of ξ_o by $\delta\gamma_{\pi\phi}$. In addition, we introduce the following expressions.

$$N^{\pi\phi} = \int_{-h}^{+h} \sigma^{\pi\phi} \, Z(\xi_o) \, d\xi_o \qquad (5.4.8)$$

$$M^{\pi\phi} = \int_{-h}^{+h} \sigma^{\pi\phi} \cdot \xi_o \cdot Z(\xi_o) \, d\xi_o \qquad (5.4.9)$$

Note that both $N^{\pi\phi}$ and $M^{\pi\phi}$ are symmetric tensors, by definition. We can now write Eq. (5.4.7) in the following from:

$$\int_V \tilde{\sigma}^{\Gamma\Lambda} \, \delta\tilde{\psi}_{\Gamma\Lambda} \, dV = \int_a [N^{\pi\phi} \, \delta\gamma_{\pi\phi} + M^{\pi\phi} \, \delta\kappa_{\pi\phi}] da \qquad (5.4.10)$$

This form is used widely in the literature and it will be adopted, herein, in order to facilitate comparison with other works.

We next proceed to deal with the second volume integral in Eq. (7.4.1). Here also we transfer all the tensors quantities to the τ-system. Vector $\bar{F}$ is transfered directly to the τ-system in a standard way by employing the shift tensor μ^{-1}, but the transfer of $\delta\tilde{v}_\Gamma$ needs some additional considerations. From the general definition for $\tilde{v}^i$, Eq. (5.3.37), we can express $\tilde{v}^i$ by:

$$\tilde{v}^i = x^i_\Lambda[v^\Lambda + \psi^\Lambda_{.\Gamma}\; y^\Gamma + \frac{Dy^\Lambda}{Dt}] \stackrel{\Delta}{=} x^i_\Lambda\; \overset{*}{v}^\Lambda \tag{5.4.11}$$

From this, it becomes clear that $\overset{*}{v}^\Lambda$ are, in fact, the components of the velocity vector $\tilde{v}$ in the τ-system. The formal transfer of $\tilde{v}_\Gamma$ to the τ-system is therefore:

$$\tilde{v}_\Gamma = \mu^\Gamma_\Lambda\; \overset{*}{v}_\Gamma \tag{5.4.12}$$

Substitution of Eq. (5.4.12) into the second integral of Eq. (5.4.1), and (similar) transfer $\bar{F}^\Gamma$ into the τ-system, making use of the orthogonality relations of the shift tensors, Eq. (5.3.62), and of Eq. (5.4.5) yield the following expression:

$$\int_V \rho\bar{B}^\Gamma\; \delta\tilde{v}_\Gamma\; dV = \int_V \rho B^\Gamma\; \delta\overset{*}{v}_\Gamma\; .\; Z(\xi_o) \cdot d\xi_o\; da \tag{5.4.13}$$

We next define two new parameters, similar to $N^{\pi\phi}$ and $M^{\pi\phi}$, by

$$b^\Gamma = \int_{-h}^{+h} \rho B^\Gamma \cdot Z(\xi_o) \cdot d\xi_o \tag{5.4.14}$$

$$d^\Gamma = \int_{-h}^{+h} \rho B^\Gamma \cdot \xi_o \cdot z(\xi_o) \cdot d\xi_o \tag{5.4.15}$$

Here, also we impose the assumption that in the expression of $\overset{*}{v}$, we can neglect all the terms of ξ_o with powers bigger than one, whenever they exist. The term $\overset{*}{\delta v}_\Gamma$ can also be separated into a linear component ξ_o with $\delta\theta_\Gamma^{(1)}$ as a coefficient and a components which is free of ξ_o, $\delta\theta_\Gamma^{(0)}$. With the help of definitions given by Eqs. (5.4.14) and (5.4.15), the second integral in Eq. (5.4.1) can be expressed as

$$\int_V \rho\tilde{B}^\Gamma \, \delta\tilde{v}_\Gamma \, dV = \int_a \left[b^\Gamma \delta\theta_\Gamma^{(0)} + d^\Gamma \delta\theta_\Gamma^{(1)}\right] da \qquad (5.4.16)$$

The treatment of the third integral in Eq. (5.4.1) is similar to the treatment of the second. Transfer of the tensorial quantities to the τ-system and substitution of Eq. (5.4.6) yields:

$$\int_A \gamma \, \tilde{F}^\Gamma \, \delta\tilde{v}_\Gamma \, dA = \int_a \gamma F^\Gamma \, \overset{*}{\delta v}_\Gamma \left[Z(\xi_o)\Big|_{\xi_o = h \pm h}\right] da \qquad (5.4.17)$$

We define now the following values:

$$f^\Gamma = \gamma F^\Gamma \left[Z(\xi_o)\Big|_{\xi_o = \pm h}\right] \qquad (5.4.18)$$

$$c^\Gamma = \gamma F^\Gamma \cdot \xi_o \cdot \left[Z(\xi_o)\Big|_{\xi_o = \pm h}\right] \qquad (5.4.19)$$

By using these definitions and separation of $\overset{*}{\delta v}$ as applied to the second integral, we obtain the following expression for the third integral:

$$\int_A \gamma\tilde{F}^\Gamma \, \delta\tilde{v}_\Gamma \, dA = \int_a \left[f^\Gamma \delta\theta_\Gamma^{(0)} + c^\Gamma \, \delta\theta_\Gamma^{(1)}\right] da \qquad (5.4.20)$$

Finally, from Eqs. (5.4.10), (5.4.16) and (5.4.20) it is possible to express the principle of Virtual Power, Eq. (5.4.1), in the following form:

$$\int_a [N^{\pi\phi}\, \delta\gamma_{\pi\phi} + M^{\pi\phi}\, \delta\kappa_{\pi\phi} - (b^\Gamma + f^\Gamma)\, \delta\theta^{(0)}_\Gamma - (d^\Gamma + c^\Gamma)\, \delta\theta^{(1)}_\Gamma]\, da = 0 \tag{5.4.21}$$

The equilibrium equations and the boundary conditions are obtaine from Eq. (5.4.21) by the application of the Gauss-Green theorem, where the character of the kinematics variables $\delta\gamma_{\pi\phi}$, $\delta\kappa_{\pi\phi}$, $\delta\theta^{(0)}_\Gamma$ and $\delta\theta^{(1)}_\Gamma$ depends on the different approximations assumptions for the deformation of the shell structure.

As an example we will derive next the equations of equilibrium and the boundary conditions for Formulation D in subsection 5.3.1; meaning, the classical theory based on the Kirchhoff-Love hypotheses. From the definition for $\psi_{\pi\phi}$ in Eq. (5.2.44) and the expressions for the shift tensors for this case (Formulation D) in Eqs. (5.3.76) and (5.3.78) we obtain:

$$\begin{aligned}\delta\gamma_{\pi\phi} = \delta\psi_{\pi\phi} &= (\delta v_{\alpha;\beta} - b_{\alpha\beta}\delta W)\, \delta^\alpha_\pi\, \delta^\beta_\phi \\ &+ (\delta W_{;\alpha} + b^\gamma_\alpha\, \delta v_\gamma)\, (\delta^3_\pi\, \delta^\alpha_\phi - \delta^\alpha_\pi\, \delta^3_\phi)\end{aligned} \tag{5.4.22}$$

and

$$\begin{aligned}\delta\kappa_{\pi\phi} = -a_{\alpha\gamma}\delta\left(\frac{Db^\gamma_\beta}{Dt}\right) \delta^\alpha_\pi\, \delta^\beta_\phi = -\{&b^\gamma_{\alpha;\beta}\, \delta v_\gamma + b^\gamma_\alpha\, \delta v_{\gamma;\beta} \\ &- b^\gamma_\beta\, \delta v_{\alpha;\gamma} + \delta W_{;\alpha\beta} + b^\gamma_\beta\, b_{\gamma\alpha}\, \delta W\}\, \delta^\alpha_\pi\, \delta^\beta_\phi\end{aligned} \tag{5.4.23}$$

The expression for $\overset{*}{v}$ in Eq. (5.4.11) reduces (for Formulation D) to:

$$\overset{*}{v}_\Gamma = [v_\alpha - \xi_o(w_{;\alpha} + b^\gamma_\alpha\, v_\gamma)]\, \delta^\alpha_\Gamma + w\delta^3_\Gamma \tag{5.4.24}$$

From the above, one may write

$$d\theta^{(0)}_{\Gamma} = (dv_\alpha)\ \delta^{\alpha}_{\Gamma} + (\delta W)\ \delta^{3}_{\Gamma} \qquad (5.4.25)$$

$$d\theta^{(1)}_{\Gamma} = -\ [\delta W_{;\alpha} + b^{\gamma}_{\alpha}\ \delta v_\gamma]\delta^{\alpha}_{\Gamma} \qquad (5.4.26)$$

Substitution of Eqs. (5.4.22), (5.4.23), (5.4.25) and (5.4.26) into the principle of Virtual Power, Eq. (5.4.21), making use of equations of the type:

$$N^{\alpha\beta}\ \delta v_{\alpha;\beta} = (N^{\alpha\beta}\delta v_\alpha)_{;\beta} - N^{\alpha\beta}{}_{;\beta}\ \delta v_\alpha \qquad (5.4.27a)$$

$$M^{\alpha\beta}\ b^{\gamma}_{\alpha}\ \delta v_{\gamma;\beta} = (M^{\alpha\beta} b^{\gamma}_{\alpha}\ \delta v_\gamma)_{;\beta} - (M^{\alpha\beta} b^{\gamma}_{\alpha})_{;\beta}\delta v_\gamma\ , \qquad (5.4.27b)$$

and using the Gauss-Green theorem, for the first term on the right hand side of Eq. (5.4.27), yields the final expression of the principle of Virtual Power for Formulation D.

$$\int_a \{[-\ N^{\alpha\beta}{}_{;\beta} - (M^{\alpha\gamma}\ b^{\beta}_{\gamma})_{;\beta} + M^{\gamma\beta}{}_{;\beta}\ b^{\alpha}_{\gamma} - (b^{\alpha} + f^{\alpha}) + (d^{\gamma} + c^{\gamma})\ b^{\alpha}_{\gamma}]\ \delta v_\alpha$$

$$+ [-N^{\alpha\beta}\ b_{\alpha\beta} - M^{\alpha\beta}{}_{;\alpha\beta} - M^{\alpha\beta}\ b^{\gamma}_{\beta}\ b_{\gamma\alpha} - (b^3 + f^3) + (d^{\gamma} + c^{\gamma})_{;\gamma}]\ \delta W\}\ a$$

$$+ \int_s \{[N^{\alpha\beta} - M^{\gamma\beta}\ b^{\alpha}_{\gamma} + M^{\gamma\alpha} + M^{\gamma\alpha}\ b^{\beta}_{\gamma}] \cdot \nu_\beta \cdot \delta v_\alpha + [M^{\alpha\beta}{}_{;\alpha} + d^{\beta} + c^{\beta}]\ \nu_\beta\ \delta W$$

$$-\ M^{\alpha\beta} \cdot \nu_\beta \cdot \delta W_{;\alpha}\}\ ds = 0. \qquad (5.4.28)$$

where ν is a unit vector normal to surface s. Thus, the equilibrium equation for the classical theory will be:

$$\left(N^{\alpha\beta} + M^{\alpha\gamma}\, b^{\beta}_{\gamma}\right)_{;\beta} - M^{\gamma\beta}{}_{;\beta}\, b^{\alpha}_{\gamma} + P^{\alpha} - m^{\gamma}\, b^{\alpha}_{\gamma} = 0 \qquad (5.4.29a)$$

$$N^{\alpha\beta}\, b_{\alpha\beta} + M^{\alpha\beta}{}_{;\alpha\beta} + M^{\alpha\beta}\, b^{\gamma}_{\beta}\, b_{\gamma\alpha} + P^{3} + m^{\gamma}{}_{;\gamma} = 0 \qquad (5.4.29)$$

$$P^{\Gamma} = b^{\Gamma} + f^{\Gamma} \quad ; \quad m^{\Gamma} = d^{\Gamma} + C^{\Gamma} \qquad (5.4.30)$$

Moreover, the boundary conditions on s from Eq. (5.4.28), under the assumption that there are no forces applied on surfaces normal to the reference; are

$$[N^{\alpha\beta} - M^{\gamma\beta}\, b^{\alpha}_{\gamma} + M^{\gamma\alpha}\, b^{\beta}_{\gamma}] \cdot \nu_{\beta} = 0 \qquad \text{or} \qquad \delta v_{\alpha} = 0 \qquad (5.4.31a)$$

$$[M^{\alpha\beta}{}_{;\alpha} + m^{\beta}] \cdot \nu_{\beta} = 0 \qquad \text{or} \qquad \delta W = 0 \qquad (5.4.31b)$$

$$M^{\alpha\beta} \cdot \nu_{\beta} = 0 \qquad \text{or} \qquad \delta W_{;\alpha} = 0 \qquad (5.4.31c)$$

Similar to the definition of ν, it is possible to define a unit vector t, tangent to s, and by this to break the partial derivative of δw into tangent to - s and normal to - s components

$$\delta W_{;\alpha} = \delta W_{;\gamma} \cdot \nu_{a} + \delta W_{;s} \cdot t_{\alpha} \qquad (5.4.32)$$

Substitution of Eq. (5.4.32) into Eq. (5.4.28) and performing integration by parts, similar to Ref. [100], yield the final shape of the boundary conditions, Eq. (5.4.31). While Eq. (5.4.31a) does not change the next two do and instead of Eqs. (5.4.31 b,c) we have:

$$(M^{\alpha\beta}{}_{;\alpha} + m^{\beta})\, \nu_{\beta} + (M^{\alpha\beta} \nu_{\beta}\, t_{\alpha})_{;s} = 0 \qquad \text{or } \delta W = 0 \tag{5.4.33a}$$

$$M^{\alpha\beta} \cdot \nu_{\beta}\, \nu_{\alpha} = 0 \qquad \text{or } \delta W_{;\alpha} = 0 \tag{5.4.33b}$$

It must be noticed here that had we not assumed that there are no external forces on the surface normal to the reference surface then, the requirement for the vanishing of the static values in the boundary conditions, Eq. (5.4.31a) and (5.4.33), would be changed to the requirement that they would be equal to the external forces.

Eqs. (5.4.29), which were obtained here for the classical theory, differ from the customary equations, as in Refs. [98] [100]. The reason for the difference lies in the definitions for $N^{\Gamma\Lambda}$ and $M^{\Gamma\Lambda}$. From Ref. 98. the stress resultant $\tilde{N}^{\Gamma\Lambda}$ and the stress couples $\tilde{M}^{\Gamma\Lambda}$ are defined in the following manner.

$$\tilde{N}^{\Gamma\Lambda} = \int_{-h}^{+h} \tilde{\sigma}^{\Gamma\Lambda} \cdot Z(\xi_o)\, d\xi_o \tag{5.4.34a}$$

$$\tilde{M}^{\Gamma\Lambda} = \int_{-h}^{+h} \tilde{\sigma}^{\Gamma\Lambda} \cdot \xi_o \cdot Z(\xi_o)\, d\xi_o \tag{5.4.34b}$$

Moreover, similar definitions for the components of the body and external forces are given. Use of these definitions yilds the following system of equations [100].

$$- N^{\alpha\beta}\, b_{\alpha\beta} + M^{\alpha\beta}\, b^{\gamma}_{\alpha}\, b_{\gamma\beta} - M^{\alpha\beta}{}_{;\alpha\beta} = 0 \tag{5.4.35a}$$

$$- N^{\alpha\beta}{}_{;\beta} - M^{\gamma\beta}\, b^{\alpha}_{\gamma;\beta} + 2(M^{\gamma\beta}\, b^{\alpha}_{\gamma})_{;\beta} = 0 \tag{5.4.35b}$$

where Eqs. (5.4.35) were written without body or external forces. These equations, Eqs (5.4.35) replace Eqs. (5.4.29).

From the definitions, given by Eqs. (5.4.34), it is possible to developed equilibrium equations for the different formulations. For example, we obtain the equilibrium equations for assumption V of subsection 5.3 by a similar process. Without external or body forces, the equilibrium equations for this case become

$$- N^{\alpha\beta}\, b_{\alpha\beta} + N^{33}\, y^{\gamma}\, y^{\delta}\, b_{\gamma\delta} + M^{\alpha\beta}\, Yb^{\gamma}_{\alpha}\, b_{\gamma\beta} - (M^{\alpha\beta}Y)_{;\alpha\beta}$$

$$+ M^{3\beta}\, y^{\gamma}[y^{\rho}\, b_{\rho\gamma;\beta} + Yb^{\delta}_{\beta}\, b_{\delta\gamma} + 2\lambda^{\delta}_{\beta}\, b_{\delta\gamma}] = 0 \qquad (5.4.36a)$$

$$[-N^{\alpha\beta} + N^{33}\, y^{\beta}\, y^{\alpha} + 2M^{\gamma\beta}\, Yb^{\alpha}_{\gamma}]_{;\beta} - M^{\gamma\beta}\, Yb^{\alpha}_{\gamma;\beta}$$

$$- \{M^{\gamma\beta}\, y^{\alpha} + \tfrac{1}{2}[M^{3\beta}\, y^{\gamma}\, y^{\alpha} + M^{3\gamma}\, y^{\beta}\, y^{\alpha}]\}_{;\beta\gamma}$$

$$+ \{M^{3\beta}[y^{\alpha}(Yb^{\gamma}_{\beta} + \lambda^{\gamma}_{\beta}) + y^{\gamma}(Yb^{\alpha}_{\beta} + \lambda^{\alpha}_{\beta})]\}_{;\gamma} = 0 \qquad (5.4.36b)$$

$$N^{\alpha 3} - [M^{\alpha\beta} + M^{3\beta}\, y^{\alpha}]_{;\beta} + M^{3\beta}[Yb^{\alpha}_{\beta} + \lambda^{\alpha}_{\beta}] = 0 \qquad (5.4.36c)$$

$$N^{33}\, Y - M^{\alpha\beta}\, b_{\alpha\beta} - [M^{3\beta}\, Y]_{;\beta} + M^{3\beta}\, [\gamma_{\beta} - y^{\gamma} b_{\gamma\beta}] = 0 \qquad (5.4.36d)$$

the related boundary conditions are:

$$[M^{\alpha\beta}Y]_{;\beta}\, \nu_{\alpha} + (M^{\alpha\beta}\, Y\nu_{\beta}\, t_{\alpha})_{;s} = 0 \qquad \text{or } \delta W = 0 \qquad (5.4.37a)$$

$$\{2M^{\gamma\beta}\, Y\, b^{\alpha}_{\gamma} + M^{3\gamma}[y^{\alpha}(Yb^{\beta}_{\gamma} + \lambda^{\beta}_{\gamma}) + y^{\beta}(Yb^{\alpha}_{\gamma} + \lambda^{\alpha}_{\gamma})]\}\, \nu_{\beta}$$

$$- \{[M^{\beta\gamma}\, y^{\alpha} + \tfrac{1}{2}\, y^{\alpha}(M^{3\beta}\, y^{\gamma} + M^{3\gamma}\, y^{\beta})]\, \nu_{\gamma}\, t_{\beta}\}_{;s} = 0 \qquad \text{or } \delta v_{\alpha} = 0$$

$$(5.4.37b)$$

$$[M^{\alpha\beta} + M^{3\beta} y^{\alpha}] \cdot \nu_{\beta} = 0 \qquad \text{or } \delta \frac{Dy_2}{Dt} = 0 \qquad (5.4.37c)$$

$$M^{3\beta} Y\nu_{\beta} = 0 \qquad \text{or } \delta \frac{DY}{Dt} = 0 \qquad (5.4.37d)$$

It will be also noticed that Eqs. (5.4.35a) and (5.4.36b) reduce for the equality to the known equations (5.4.35), as expected. Similarly, the boundary conditions, Eqs. (5.4.37a) and (5.4.37b), reduce to those found in Ref. [100].

5.4.2 - The Principle of the Rate of Virtual Power and the Rate of Equations of Equilibrium

The principle of the rate of virtual power is obtained by total differentiation of the principle of virtual power [see Eq. (5.4.1)],

$$\frac{d}{dt} \int_V \bar{\sigma}^{\Gamma\Lambda}\, \delta\bar{\psi}_{\Gamma\Lambda}\, dV - \frac{d}{dt} \int_V \rho\, \bar{B}^{\Gamma}\, \delta\bar{v}_{\Gamma}\, dV - \frac{d}{dt} \int_A \gamma\, \bar{F}^{\Gamma}\, \delta\bar{v}_{\Gamma}\, dA = 0 \qquad (5.4.38)$$

There is a need to perform a formal time differentiation of every one of the integrals in Eq. (5.4.38). Substitution of the definitions given by Eqs. (5.4.8) and (5.4.9), into the first integral in Eq. (5.4.38) yields,

$$I \stackrel{\Delta}{=} \frac{d}{dt} \int_V \bar{\sigma}^{\Gamma\Lambda}\, \delta\bar{\psi}_{\Gamma\Lambda}\, dV = \frac{d}{dt} \int_a [N^{\Gamma\Lambda} \delta\gamma_{\Gamma\Lambda} + M^{\Gamma\Lambda}\, \delta\kappa_{\Gamma\Lambda}]\, da$$

or,

$$I = \int_a [\frac{dN^{\Gamma\Lambda}}{dt}\, \delta\gamma_{\Gamma\Lambda} + \frac{dM^{\Gamma\Lambda}}{dt}\, \delta\kappa_{\Gamma\Lambda} + N^{\Gamma\Lambda}\, \frac{d(\delta\gamma_{\Gamma\Lambda})}{dt} + M^{\Gamma\Lambda}\, \frac{d(\delta\kappa_{\Gamma\Lambda})}{dt}]\, da$$
$$+ \int_a [N^{\Gamma\Lambda}\, \delta\gamma_{\Gamma\Lambda} + M^{\Gamma\Lambda}\, \delta\kappa_{\Gamma\Lambda}]\, \frac{d(da)}{dt} \qquad (5.4.40)$$

where the time derivative of an elmental area of the surface is from Eq. (5.4.5):

$$\frac{d(da)}{dt} = \frac{d}{dt}\left(\sqrt{g}\right) d\theta^1 \, d\theta^2 = \psi^{\Omega}_{.\Omega} \, da \tag{5.4.41}$$

Substitution of Eq. (5.4.41) into Eq. (5.4.40) yields the following expression for I:

$$I = \int_a \Big\{ \Big[\frac{dN^{\Gamma\Lambda}}{dt} + N^{\Gamma\Lambda} \, \psi^{\Omega}_{.\Omega}\Big] \, \delta\gamma_{\Gamma\Lambda} + \Big[\frac{d\mu^{\Gamma\Lambda}}{dt} + M^{\Gamma\Lambda} \, \psi^{\Omega}_{.\Omega}\Big] \, \delta\kappa_{\Gamma\Lambda}$$

$$+ \left[N^{\Gamma\Lambda}\right] \frac{d(\delta\gamma_{\Gamma\Lambda})}{dt} + \left[M^{\Gamma\Lambda}\right] \frac{d(\delta\kappa_{\Gamma\Lambda})}{dt} \Big\} \, da \tag{5.4.42}$$

The second integral in Eq. (5.4.38) can be developed in a similar manner. By using the definitions, given by Eqs. (5.4.13) and (5.4.14), we obtain [see Eq. (5.4.16)]:

$$J_1 \stackrel{\Delta}{=} \frac{d}{dt} \int_a \left[b^{\Gamma} \, \delta\theta^{(0)}_{\Gamma} + d^{\Gamma} \, \delta\theta^{(1)}_{\Gamma}\right] da \tag{5.4.43}$$

Finally, for the third integral in Eq. (5.4.38) we obtain from Eqs. (5.4.18) and (5.4.19):

$$J_2 \stackrel{\Delta}{=} \frac{d}{dt} \int_a \left[f^{\Gamma} \, \delta\theta^{(0)}_{\Gamma} + c^{\Gamma} \, \delta\theta^{(1)}_{\Gamma}\right] da \tag{5.4.44}$$

Use of Eq. (5.4.41) into Eqs. (5.4.43) and (5.4.44), the addition of these two and use of definition, given by Eq. (5.4.30), finally yields:

$$J \stackrel{\Delta}{=} J_1 + J_2 = \int_a \Big\{ \Big[\frac{dP^{\Gamma}}{dt} + P^{\Gamma} \, \psi^{\Omega}_{.\Omega}\Big] \, \delta\theta^{(0)}_{\Gamma} + \Big[\frac{dm^{\Gamma}}{dt} + m^{\Gamma} \, \psi^{\Omega}_{.\Omega}\Big] \, \delta\theta^{(1)}_{\Gamma}$$

$$+ \left[P^{\Gamma}\right] \frac{d(\delta\theta^{(0)}_{\Gamma})}{dt} + \left[m^{\Gamma}\right] \frac{d(\delta\theta^{(1)}_{\Gamma})}{dt} \Big\} \, da \tag{5.4.45}$$

Hence, the principle of the rate of virtual power, generally, can be expressed by:

$$I - J = 0 \tag{5.4.46}$$

where I express the rate of the internal stresses power and J the rate of the external (applied forces) power.

Because of the nature of the elasto-visco-plastic constitutive relations, there is a need to replace the total time derivatives in Eq. (5.4.42), by Jaumann derivatives. From Eq. (3.2.51) we obtain for I:

$$I = \int_a \Big\{ \Big[\frac{d^J N^{\Gamma\Lambda}}{dt} + \omega^{\Gamma}_{.\phi} N^{\phi\Lambda} + \omega^{\Lambda}_{.\phi} N^{\Gamma\phi} + N^{\Gamma\Lambda} \psi^{\Omega}_{.\Omega}\Big] \delta\gamma_{\Gamma\Lambda}$$

$$+ \Big[\frac{d^J M^{\Gamma\Lambda}}{dt} + \omega^{\Gamma}_{.\phi} M^{\phi\Lambda} + \omega^{\Lambda}_{.\phi} M^{\Gamma\phi} + M^{\Gamma\Lambda} \psi^{\Omega}_{.\Omega}\Big] \delta\kappa_{\Gamma\Lambda}$$

$$+ [N^{\Gamma\Lambda}] \frac{d(\delta\gamma_{\Gamma\Lambda})}{dt} + [M^{\Gamma\Lambda}] \frac{d(\delta\kappa_{\Gamma\Lambda})}{dt} \Big\} \, da \tag{5.4.47}$$

As an example we shall develop now the principle of the rate of virtual power for the classical theory of shells, Formulation D in subection 5.3. For the sake of simplicity, we assume that there are no external forces and hence, the development deals only with the explicit expression for the internal power, I. It must be noticed that the addition of the integral which expresses, the influence of the external forces is not causing any difficulty at all. In order to obtain the explicit expression for I in Eq. (5.4.47), there is need to develop the expressions for the total time derivatives of $\delta\gamma_{\Gamma\Lambda}$ and $\delta\kappa_{\Gamma\Lambda}$. Derivation according to time of Eqs. (5.4.22) and (5.4.23) yields:

$$\frac{d(\delta\gamma_{\Gamma\Lambda})}{dt} = \Big\{\frac{d(\delta v_\alpha)}{dt}_{;\beta} - b_{\alpha\beta} \frac{d(\delta w)}{dt} - \psi^{\gamma}_{.\beta} \, \delta v_{\alpha;\gamma}$$

$$+ [\psi^{\Gamma}_{.\beta} b_{\alpha\gamma} - \rho_{\alpha;\beta}] \, \delta W - [{}_{\alpha}{}^{\gamma}{}_{\beta}] \, \delta v_\gamma\Big\} \, \delta^{\alpha}_{\Gamma} \, \delta^{\beta}_{\Lambda}$$

$$+ \left\{- b_\alpha^\gamma \; \frac{d(\delta v_\gamma)}{dt} - \frac{d(\delta w)}{dt}_{;\alpha} + \rho^\gamma \; \delta v_{\alpha;\gamma}\right.$$

$$\left. - [\rho^\gamma_{;\alpha} - \psi^\delta_{.\alpha} \; b^\gamma_\delta] \; \delta v_\gamma - \rho^\gamma \; b_{\alpha\gamma} \; \delta W + \psi^\gamma_{.\alpha} \; \delta W_{;\gamma}\right\} \delta^\alpha_\Gamma \; \delta^3_\Lambda$$

$$+ \left\{b^\gamma_\beta \frac{d(\delta v_\gamma)}{dt} + \frac{d(\delta w)}{dt}_{;\beta} - \psi^\delta_{.\beta} \; b^\gamma_\delta \; \delta v_\gamma\right.$$

$$\left. - \rho^\gamma \; b_{\beta\gamma}\delta W - \psi^\gamma_{.\beta} \; \delta W_{;\gamma}\right\} \delta^3_\Gamma \; \delta^\beta_\Lambda \qquad (5.4.48)$$

$$\frac{d(\delta\kappa_{\Gamma\Lambda})}{dt} = - \left\{b^\gamma_{\alpha;\beta} \frac{d(\delta v_\gamma)}{dt} + b^\gamma_\alpha \frac{d(\delta v_\gamma)}{dt}_{;\beta} - b^\gamma_\beta \frac{d(\delta v_\alpha)}{dt}_{;\gamma}\right.$$

$$+ \frac{d(\delta w)}{dt}_{;\alpha\beta} + b^\gamma_\beta \; b_{\gamma\alpha} \frac{d(\delta w)}{dt} + [\rho^\gamma_{;\alpha\beta} - (\psi^\delta_{.\alpha} \; b^\gamma_\delta)_{;\beta}$$

$$- \psi^\delta_{.\beta} \; b^\gamma_{\alpha;\delta} + \left[{}^{\;\gamma}_{\alpha\;\delta}\right] b^\delta_\beta + \left[{}^{\;\delta}_{\alpha\;\beta}\right] b^\gamma_\delta] \; \delta v_\gamma$$

$$+ [\rho^\gamma_{;\alpha} - \psi^\delta_{.\alpha} \; b^\gamma_\delta] \; \delta v_{\gamma;\beta} - [\psi^\delta{}_\beta \; b^\gamma_\alpha] \; \delta v_{\gamma;\delta}$$

$$- [\rho^\gamma_{;\beta} - \psi^\delta_{.\beta} \; b^\gamma_\delta - \psi^\gamma_\delta \; b^\delta_\beta] \; \delta v_{\alpha;\gamma}$$

$$+ [b_{\gamma\alpha}(\rho^\gamma_{;\beta} - \psi^\delta_{.\beta} \; b^\gamma_\delta) + b_{\gamma\beta} \; (\rho^\gamma_{;\alpha} - \psi^\delta_{.\alpha} \; b^\gamma_\delta)] \; \delta W$$

$$\left. - \left[\frac{D\gamma^\gamma_{\alpha\beta}}{Dt}\right] \delta W_{;\gamma} - [\psi^\gamma_{.\alpha}] \; \delta W_{;\gamma\beta} - [\psi^\gamma_{.\beta}] \; \delta W_{;\alpha\gamma}\right\} \delta^\alpha_\Gamma \; \delta^\beta_\Lambda$$

$$- \left\{\rho^\delta \; b^\gamma_{\beta;\delta} \; \delta v_\gamma + b^\gamma_\alpha \; \rho^\delta \; \delta v_{\gamma;\delta} - b^\gamma_\delta \; \rho^\delta \; \delta v_{\alpha;\gamma} + \rho^\delta \; b^\gamma_\delta \; b_{\gamma\alpha} \; \delta w\right.$$

$$\left. + \rho^\gamma \; \delta W_{;\alpha\gamma}\right\} \delta^\alpha_\Gamma \; \delta^3_\Lambda - \left\{[\rho^\delta \; b^\gamma_{\delta;\beta} + \rho^\gamma_{;\delta} b^\delta_\beta] \; \delta v_\gamma\right.$$

$$\left. + \rho^\delta \; b^\gamma_\delta \; \delta v_{\gamma;\beta} + \rho^\delta b^\gamma_\delta \; b_{\gamma\beta} \; dW + \rho^\gamma \; \delta W_{;\gamma\beta}\right\} \delta^3_\Gamma \; \delta^\beta_\Lambda \qquad (5.4.49)$$

where we made use of the connections between the total time derivatives of covariant derivatives and the covariant derivatives of the total time one (see section 2). We define also the following symbol:

$$[{}_{\alpha}{}^{\gamma}{}_{\beta}] \overset{\Delta}{=} \frac{D\gamma_{\alpha\beta}}{Dt} - \psi^{\gamma}_{.\alpha;\beta} \tag{5.4.50}$$

Substitution of Eqs. (5.4.22), (5.4.23), (5.4.48) and (5.4.49) into Eq. (5.4.48) and reordering of terms yield the explicit expression for I:

$$\begin{aligned}
I = \int_V & \Big\{\Big\{\frac{D^J N^{\gamma\alpha}}{dt} - \frac{d^J M^{\alpha\beta}}{dt} b^{\gamma}_{\beta} + \frac{d^\tau M^{\gamma\beta}}{dt} b^{\alpha}_{\beta} + \omega^{\gamma}_{.\delta} N^{\delta\alpha} - d^{\alpha}_{.\delta} N^{\gamma\delta} \\
& + \psi^{\delta}_{.\delta} N^{\gamma\alpha} - M^{\alpha\beta} \rho^{\gamma}_{;\beta} + M^{\gamma\beta} \rho^{\alpha}_{;\beta} - M^{\gamma\beta}[\psi^{\gamma}_{.\delta} b^{\delta}_{\beta}] + M^{\delta\beta}[\psi^{\gamma}_{.\delta} b^{\alpha}_{\beta}] \\
& - M^{\alpha\beta}[\psi^{\delta}_{.\delta} b^{\gamma}_{\beta}] + M^{\gamma\beta}[\psi^{\delta}_{.\delta} b^{\alpha}_{\beta}] + M^{\alpha\delta}[d^{\beta}_{.\delta} b^{\gamma}_{\beta}] + M^{\delta\beta}[d^{\alpha}_{.\delta} b^{\gamma}_{\beta} - d^{\gamma}_{.\delta} b^{\alpha}_{\beta}] \\
& - M^{\gamma\delta}[d^{\beta}_{\delta} b^{\alpha}_{\beta}] - \rho^{\gamma}[N^{3\alpha} + M^{3\beta} b^{\alpha}_{\beta}]\Big\} \delta v_{\gamma;\alpha} - \Big\{\frac{d^J M^{\alpha\beta}}{dt} b^{\gamma}_{\alpha;\beta} \\
& + N^{\alpha\beta}[{}_{\alpha}{}^{\gamma}{}_{\beta}] + \rho^{\gamma}_{;\alpha}[N^{\alpha 3} + M^{\delta 3} b^{\alpha}_{\delta}] + M^{\alpha\beta}[\rho^{\gamma}_{;\alpha\beta} - \psi^{\delta}_{.\alpha;\beta} b^{\gamma}_{\delta} - b^{\gamma}_{\delta}[{}_{\alpha}{}^{\delta}{}_{\beta}] \\
& + b^{\delta}_{\beta}[{}_{\alpha}{}^{\gamma}{}_{\beta}] + \psi^{\delta}_{.\delta} b^{\gamma}_{\alpha;\beta}] - 2M^{\delta\beta}[d^{\alpha}_{.\beta} b^{\gamma}_{\alpha;\beta}]\Big\} \delta v_{\gamma} - \Big\{\frac{d^J N^{\alpha\beta}}{dt} b_{\alpha\beta} \\
& + \frac{d^J M^{\alpha\beta}}{dt} b^{\gamma}_{\beta} b_{\gamma\alpha} + N^{\alpha\beta}[\rho_{\alpha;\beta} + \psi^{\gamma}_{.\alpha} b_{\gamma\beta} + \psi^{\gamma}_{.\gamma} b_{\alpha\beta} - 2d^{\gamma}_{.\alpha} b_{\gamma\beta}] \\
& + 2M^{\alpha\beta}[\rho^{\gamma}_{;\alpha} b_{\gamma\beta} + d^{\delta}_{\alpha} b^{\gamma}_{\beta} b_{\gamma\delta} + \psi^{\delta}_{.\delta} b^{\gamma}_{\beta} b_{\gamma\alpha}]\Big\} \delta W + \Big\{M^{\alpha\beta} \Big[2d_{\delta\alpha;\beta} \\
& - d_{\alpha\beta;\delta}\Big] a^{\gamma\delta}\Big\} \delta W_{;\gamma} - \Big\{\frac{d^J M^{\alpha\gamma}}{dt} + M^{\alpha\gamma} \psi^{\gamma}_{.\delta} - 2M^{\alpha\beta} d^{\gamma}_{\beta}\Big\} \delta W_{;\alpha\beta}\Big\} dV
\end{aligned}$$

$$+\int_V \{N^{\alpha\beta}[\frac{d(\delta v_\alpha)}{dt}_{;\beta} - b_{\alpha\beta}\frac{\delta(\delta W)}{dt}] - M^{\alpha\beta}[b^\gamma_{\alpha;\beta}\frac{d(\delta v_\gamma)}{dt}$$

$$+ b^\gamma_\alpha \frac{d(\delta v_\gamma)}{dt}_{;\beta} - b^\gamma_\beta \frac{d(\delta v_\alpha)}{dt}_{;\gamma} + \frac{d(\delta W)}{dt}_{;\alpha\beta} + b^\gamma_\beta\, b_{\gamma\alpha}\frac{d(\delta W)}{dt}\}\, dV \tag{5.4.51}$$

As already mentioned, it is relatively easy to obtain the explicit expression for J, in Eq. (5..45), and thus complete the formulation of the principle of the rate of virtual power for the classical case of Formulation D). It must be noticed that the second integral in Eq. (5.4.51) is equal to the principle of virtual power itself and therefore can be erased from the expression for I.

The principle of the rate of the virtual power is equivalent by definition to the equations of the rate of equilibrium and the corresponding boundary conditions. The equations of the rate of equilibrium can be developed therefore from eq. (5.4.51) by the same method we used to obtain the equation of equlibrium Eq. (5.4.29), from the principle of virtual power or by direct time differentiation of Eq. (5.4.29). In both case we obtain the following system of equations:

$$[\frac{d^J N^{\alpha\beta}}{dt} + \frac{d^T M^{\alpha\gamma}}{dt} b^\beta_\gamma]_{;\beta} - \frac{d^J M^{\gamma\beta}}{dt}_{;\beta} b^\alpha_\gamma + N^{\alpha\beta}[{}_\gamma{}^\alpha{}_\beta] + N^{\alpha\beta}\,[\psi^\gamma_{.\gamma;\beta}]$$

$$+ [\omega^\alpha_{.\gamma}\, N^{\gamma\beta}]_{;\beta} - [d^\beta_{.\gamma}\, N^{\gamma\alpha}]_{;\beta} + M^{\gamma\beta}[b^\delta_\beta\,[{}_\gamma{}^\alpha{}_\delta] - b^\alpha_\delta\,[{}_\gamma{}^\delta{}_\beta]]$$

$$+ M^{\alpha\delta}\{[\psi^\gamma_{.\delta}\, b^\beta_\gamma]_{;\beta} - [\psi^\gamma_{.\beta}\, b^\beta_\delta]_{;\gamma}\} - M^{\alpha\beta}{}_{;\gamma}\,[\psi^\gamma_{.\delta}\, b^\delta_\beta - \rho^\gamma_{;\beta}]$$

$$+ M^{\alpha\beta}{}_{;\gamma}\,[\psi^\delta_{.\beta} b^\alpha_\gamma - \rho^\alpha_{;\beta}] + \{w^\alpha_{.\delta}\, M^{\delta\gamma} - d^\gamma_{.\delta}\, M^{\alpha\delta}]\, b^\beta_\gamma\}_{;\beta} - [w^\gamma_{.\delta}\, M^{\delta\beta}$$

$$- d^{\beta}_{.\delta}\ M^{\gamma\delta}]_{;\beta}\ b^{\alpha}_{\gamma} + M^{\alpha\gamma}\ \psi^{\beta}_{.\beta;\delta} b^{\delta}_{\gamma} - M^{\gamma\delta}\ \psi^{\beta}_{.\beta;\delta}\ b^{\alpha}_{\gamma}$$

$$- \rho^{\alpha}[N^{3\beta} + M^{3\gamma}\ b^{\beta}_{\gamma}]_{;\beta} = 0 \tag{5.4.52a}$$

$$\frac{d^J N^{\alpha\beta}}{dt}\ b_{\alpha\beta} + \frac{d^J M^{\alpha\beta}}{dt}{}_{;\alpha\beta} + \frac{d^J M^{\alpha\beta}}{dt}\ b^{\gamma}_{\beta}\ b_{\gamma\beta} + N^{\alpha\beta}\ \{\rho_{\alpha;\beta}$$

$$+ b_{\gamma\alpha}\ \psi^{\gamma}_{.\beta} - 2d^{\gamma}_{.\alpha}\ b_{\gamma\beta}\} + M^{\alpha\beta}\{\psi^{\gamma}_{.\gamma;\alpha\beta} + [{}_{\alpha}{}^{\gamma}{}_{\beta}]_{;\gamma}$$

$$+ 2\rho^{\gamma}_{.\alpha}\ b_{\gamma\beta} - d^{\gamma}_{\alpha;\gamma\beta} + w^{\gamma}_{.a;\beta\gamma}\} - M^{\alpha\beta}{}_{;\gamma}\ a^{\gamma\delta}\ d_{\alpha\beta;\delta}$$

$$+ 2M^{\alpha\beta}{}_{;\beta}\ \{\psi^{\gamma}_{.\gamma;\alpha} - d^{\gamma}_{\alpha;\gamma}\} - \{M^{\alpha\beta}{}_{;\gamma\beta} + \mu^{\alpha\beta}{}_{;\beta\gamma}\}\ d^{\gamma}_{\alpha} = 0 \tag{5.4.52b}$$

It might be thought that in the principle itself, Eq. (5.4.41), and in the equations of equilibrium, Eqs. (5.4.52) also, there exist additional terms which depend on $N^{3\alpha}$and $M^{3\alpha}$. The requirement of symmetry of the Cauchy stress tensor in the system (~):

$$\tilde{\varepsilon}_{\Gamma\Lambda\pi}\tilde{\sigma}^{\Gamma\Lambda} = 0\ , \tag{5.4.53}$$

can be transfer to the τ-system in the following manner:

$$\tilde{\varepsilon}_{\Gamma\Lambda\pi}\ \tilde{\sigma}^{\Gamma\Lambda} = Z(\xi_o)\ \varepsilon_{\Gamma\Lambda\pi}\ .\ \sigma^{\phi\Omega}\ \mu^{-1\Gamma}_{\ \ \phi}\ \mu^{-1\Lambda}_{\ \ \Omega} \tag{5.4.54}$$

Integration of Eq. (5.4.54) through the thickness of the shell and use of the definitions, given by Eqs (5.4.8) and (5.4.9), yield, among others, the following condition (for $\Gamma = 3$):

$$\epsilon_{3\gamma\alpha}\{N^{3\gamma} + M^{3\delta}b_{\delta}^{\gamma}\} = 0 \tag{5.4.55}$$

From this it is clear that the terms in the brackets of Eq. (5.4.55) must be zero themselves without any connection to the permutation tensor. Study of Eqs. (5.4.51) and (5.4.52) shows that substitution of Eq.(5.4.55) will lead to the absence of $N^{\alpha 3}$ and $M^{\alpha 3}$ from several terms (simplification).

6. PLANAR CURVED BEAMS

6.1 Introduction

Section 6 deals with the kinematic and the constitutive equations used for the numerical analysis of planar curved beams.

Formulation of problems concerned with finite deformation of beams has followed two different paths[101]. Prescribing the beam by its deformed or undeformed centroidal axis and cross section, one may introduce at the outset beam stress resultants and their conjugate kinematic variables characterizing displacement and rotation of the cross section. Together with appropriate beam constitutive equations and a global balance law a consistent theory is obtained. Alternatively, one may imbed beam theory in the setting of deformable solid continua, in which case one is concerned with local constitutive equations connecting the stress tensor with a strain tensor, which may in turn be expressed in terms of a combination of undetermined beam kinematic variables and functions of the beam coordinates. Momentum may then be balanced globally by integrating the local equations over the deformed beam configuration. Both paths will be considered in what follows.

A complete abinito rate theory for the second path can be obtained by an appropriate plane stress approximation of the three-dimensional formulations presented in sections two and three. This approach is presented in Subsection 6.2. The rate form of the field equations for finite strains and rotations of curved beams according to the first path, can be obtained by a careful reduction of the shell theory, presented in section five. A simplified version of this formulation is presentd in Subsection 6.3. Finally, three simple numerical examples are demonstrated in Subsection 6.4.

6.2 Plane Stress Approximation

The constitutive relation presented in section three will be rephrased here, in some different and compact form as follows

a) if $$F = (t^i_k - C\,\rho_o\,g\,\beta^i_k)(t^k_i - C\,\rho_o\,g\,\beta^k_i) - k^2(\overset{P}{W},T) = 0 \qquad (6.2.1)$$

where, s^i_k, the Kirchhoff stress tensor, $s^i_k = \frac{\rho_o}{\rho}\,\sigma^i_k$, and the temperature T are independent process variables, t^i_k being the deviator of the Kirchhoff stress. s^i_k, and $\overset{P}{W}$ and β^i_k are internal parameters, then

$$d^i_k = \underbrace{\frac{1}{2Gv}\left\{\overset{\nabla}{s}{}^i_k - \frac{\nu}{1+\nu}\,\overset{\nabla}{s}{}^r_r\,\delta^i_k\right\} + \alpha\,\dot{T}\,\delta^i_k}_{\overset{E}{d}{}^i_k} + \underbrace{2\dot{\lambda}(\bar{t}^i_k - C\,\rho_o\,g\,\beta^i_k)}_{\overset{P}{d}{}^i_k} \qquad (6.2.2)$$

with

$$\dot{\lambda} = \frac{1}{4n}\left\{\left[\frac{(t^i_k - C\,\rho_o\,g\,\beta^i_k)(t^k_i - C\,\rho_o\,g\,\beta^k_i)}{k^2}\right]^{1/2} - 1\right\}$$

$$\bar{t}^i_k = \frac{1}{1 + 4\,n\dot{\lambda}}\,(t^i_k - C\,\rho_o\,g\,\beta^i_k) + C\,\rho_o\,g\,\beta^i_k \qquad (6.2.3)$$

$$\dot{\overset{P}{W}} = \frac{1}{\rho_o}\,\bar{t}^i_k\,\overset{P}{d}{}^k_i \qquad (6.2.4)$$

$$\overset{\nabla}{\beta}{}^i_k = \zeta\,\overset{P}{d}{}^i_k \qquad (6.2.5)$$

b) if $$F = 0 \qquad (6.2.6)$$

and $$\frac{\partial F}{\partial s^i_k} \overset{\nabla}{s}{}^i_k + \frac{\partial F}{\partial T} \dot{T} > 0 \tag{6.2.7}$$

then $$d^i_k = \overset{E}{d}{}^i_k \tag{6.2.8}$$

$$\overset{P}{d}{}^i_k = 0 \qquad \overset{\overset{P}{\nabla}}{d}{}^i_k = 2\ddot{\lambda}(t^i_k - C\rho_o \, g\beta^i_k) \tag{6.2.9}$$

with $$\ddot{\lambda} = \frac{1}{8nk^2} \{2(t^i_k - C \, \rho_o \, g \, \beta^i_k) \overset{\nabla}{t}{}^k_i - \frac{\partial k^2}{\partial T} \dot{T}\} \tag{6.2.10}$$

c) if $$F = 0 \quad \text{and} \quad \frac{\partial F}{\partial S^i_k} \overset{\nabla}{s}{}^i_k + \frac{\partial F}{\partial T} \dot{T} \leq 0 \tag{6.2.11}$$

or $$F < 0 \tag{6.2.12}$$

then $$d^i_k = \overset{E}{d}{}^i_k \tag{6.2.13}$$

$$\overset{P}{\dot{W}} = 0 \tag{6.2.14}$$

$$\overset{\nabla}{\beta}{}^i_k = 0 \tag{6.2.15}$$

By definition, a body is said to be in the state of plane stress parallel to the u^1, u^2 plane when the stress components σ^{13}, σ^{23}, σ^{33} vanish[102]. It is well known in literature that the case of plane stress isdifficult to handle theoretically. Even linear elasticity has to treat this case in an approximate manner. To remove some of theoretical difficulties Durban and Baruch[103] introduced the notion of Generalized Plane Stress, where instead of dealing with the quantities themselves, one deals with their average values.

In our case the problem is even more difficult. The nonlinearities, which the general three-dimensional theory takes into account will also cause a large change of the geometrical quantities in the u^3 direction. Clearly, some assumptions are needed to treat the case of plane stress as a two-dimensional case.

The first basic assumption is that the thickness, h, of the plate defined by the coordinates u^1, u^2 located in its middle plane, is small as compared with the other two dimensions. A second assumption is that the external forces act in the u^1, u^2 directions and are symmetrically distributed with respect to the middle plane.

In a way similar to the procedure proposed by Durban and Baruch[103], all the kinematic expressions are obtained by averaging the three-dimensional expressions.

A basic assumption for the case of plane stress is that the components connected with the third direction are small and can be neglected. So, a new concept of generalized stress tensor is introduced

$$\tau^i_k = \frac{\sigma^i_k h}{h_o} \tag{6.2.16}$$

It must be noted that in the linear theory of elasticity, where the geometry does not change, the averaged and generalized stress tensors coincide.

So the three-dimensional incremental elasto-viscoplastic theory, developed previously, can be adopted for two-dimension plane stress problems.

6.3 A Simplified Version of Thin Curved Beam Element

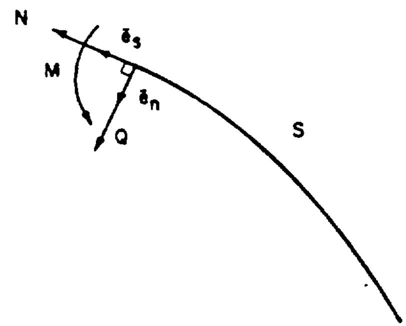

Fig. 6.1 - Reference Line of a Curved Beam

A portion of the reference line for a curved beam is shown on Fig. 6.1. The current arc length is denoted by s, while ϕ is the current angle of inclination of the normar to the reference line, and ρ is the radius of curvature.

The stress resultants acting on the beam cross section are the bending moment M, the axial force N, and the shear force Q. The external load, measured per unit of current length of the reference line, has the components p_s and p_n in the direction of the unit vectors $\bar{e}_s$ and $\bar{e}_n$ respectively.

If v_s and v_n denote the velocity components in the direction of the unit vectors $\bar{e}_s$ and $\bar{e}_n$ respectively, the rate of extension is

$$d = \frac{\partial v_s}{\partial s} - \frac{v_n}{\rho} \tag{6.3.1}$$

The rate of rotation, $\dot{\phi}$, of a given section is given by

$$\omega = \dot{\phi} = \frac{\partial v_n}{\partial s} + \frac{v_s}{\rho} \tag{6.3.2}$$

while the generalized rate of deformation associated with bending is

$$\dot{k} = \frac{\partial \dot{\phi}}{\partial s} = -\frac{\partial}{\partial s}\left(\frac{\partial v_n}{\partial s} + \frac{v_s}{\rho}\right) \tag{6.3.3}$$

The rate of equilibrium equations for this simplified version may be put in the following form:

$$\begin{aligned}
&\frac{\partial \dot{N}}{\partial s} - Q\,\frac{\partial \phi}{\partial s} - \dot{Q}\,\frac{\partial \phi}{\partial S} + \dot{p}_s + d\;p_s = 0 \\
&\frac{\partial \dot{Q}}{\partial s} + N\,\frac{\partial \dot{\phi}}{\partial s} + \dot{N}\,\frac{\partial \phi}{\partial s} + \dot{p}_n + d\;p_n = 0 \\
&\frac{\partial \dot{M}}{\partial s} + \dot{Q} + dQ = 0
\end{aligned} \tag{6.3.4}$$

6.4 Numerical Examples

The quasi-linear nature of the velocity equilibrium equations suggests the adoption of an incremental approach to numerical integration with respect to time. The availability of the field formulation provides assurance of the completeness of the incremental equations and allows the use of any convenient procedure for spatial integration over the domain B. In the present instance the choice has been made in favor of a simple first order expansion in time for the construction of incremental solutions from the results of finite element spatial integration of the governing equations.

The procedure employed permits the rates of the field formulation to be interpreted as increments in the numerical solution. This is particularly convenient for the construction of incremental boundary condition histories.

The capabilities of the presented models here-in have been evaluated through three simple numerical examples. The first example demonstrates the capability of the plane stress approximation to predict deflections and stresses in a beam loaded by a constant moment. Figure 6.2 illustrates the beam and the finite element model. A quarter of the beam was divided into six elements in the vertical direction and into five elements in the horizontal direction. The external moment was introduced by six parallel forces acting on the section BC (see Fig. 6.2).

The value of the external moment is 3500 kg/cm, and the material of the beam is CHR-17. The viscoplastic properties of the material were obtained experimentally from uniaxial tests in Ref. 9. This properties were collaborated into the present material model.

The variation of the deflection of point E as a function of time is given in Fig. 6.3. It is important to point out the value of the large deformation analysis. After ten minutes of the deformation is increased by 42% and at the same time there are important changes in the stress field (see Fig. 6.4).

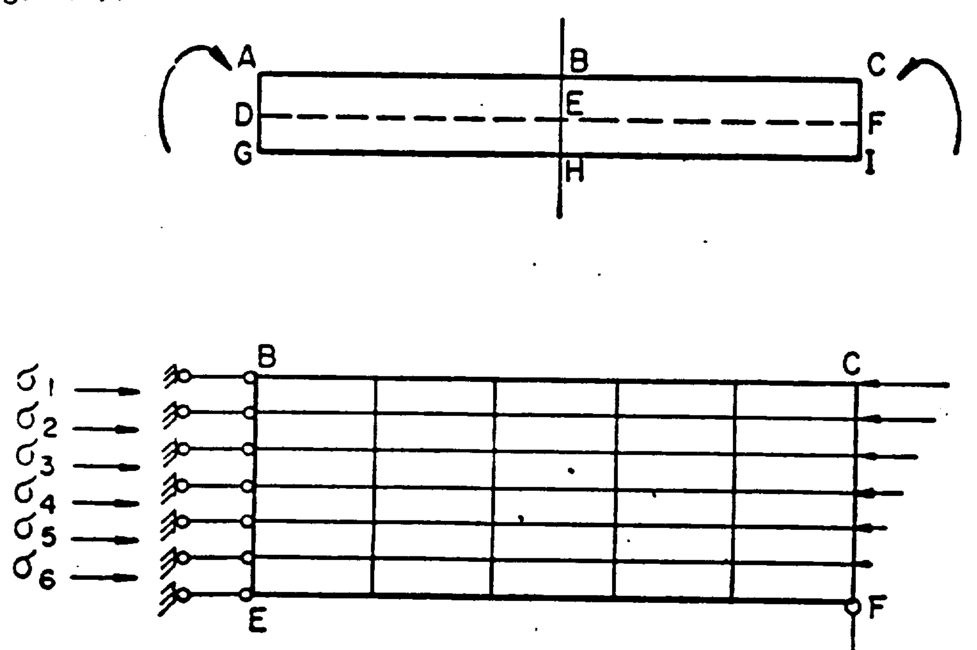

Fig. 6.2 - The Beam Model

The next example consists of a straight simply supported beam, loaded by a transverse concentrate force at the midspan. The beam is 25 inches long, two inches high and one inch wide. The material is stainless steel 304 (Heat 9T2796). The material constants in sub section 6.2 were correlated with the uniaxial tension experimental results given in Ref.104. The beam was subjected to a load of 2000 pounds at 1100^{O}F, this load was then held constant for 312 hr., and then increased to 2250 pounds at 1400^{O}F.

The primary purpose of this example is to compare the results, obtained by the two previously discussed models. The first one is the two-dimensional plane stress model, and the second one is the thin beam model as derived from thin shell theory. Figure 6.5 presents results in the form of load versus midspan deflection. The finite element model consists of five simple plane stress elements (dashed line in Fig. 6.5) or five sophisticated beam elements (full line in Fig. 6.5).

It can be seen (Fig. 6.5) that the results agree quite well up to the 312-hour hold period (points 3,4). During the hold period, the material hardens and only the beam model can represent this behavior after the load is further increased.

The last example presents an analysis of a circular arch. The geometry of the shallow circular arch is shown on Fig. 6.6. The material is once again the 304 stainless steel. The arch is fixed at both ends and carries a concentrated load at the center. The elasto-viscoplastic analysis of this arch is performed with the aid of a ten curved beam elment model and with the inertia terms taken into account. The load P is assumed to be applied in a quasi-static manner at t = 0. The results of this analysis are shown on Fig. 6.6, as the time-history of the midspan displacement. The response of the arch starts with the instantaneous elastic deformation at t = 0, followed by slow deformation up to point B, which can be considered as a limit point for the given value of the load P. Beyond point B, the displacements increase rapidly towards point C. This may suggest the existence of critical time for the prescribed load.

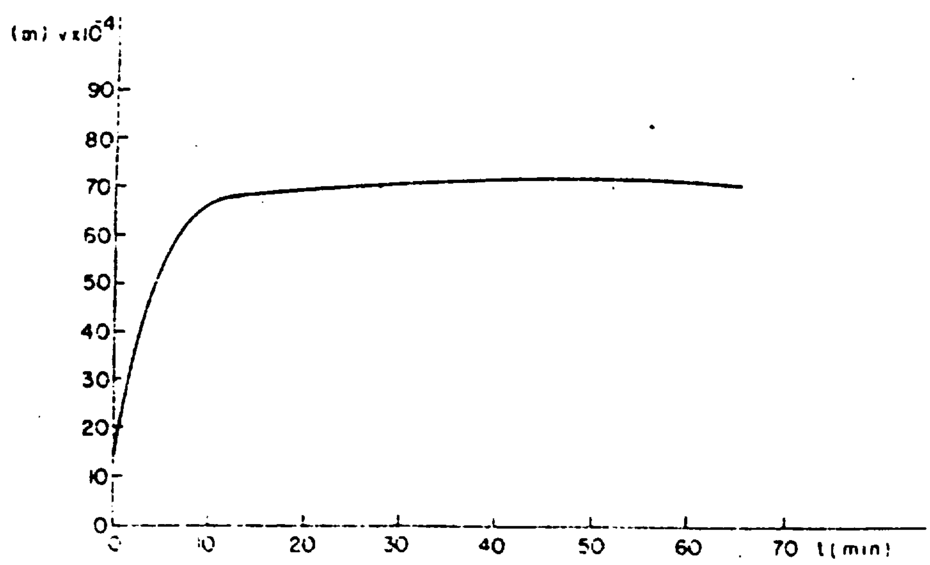

Fig. 6.3 - Point E Deflection

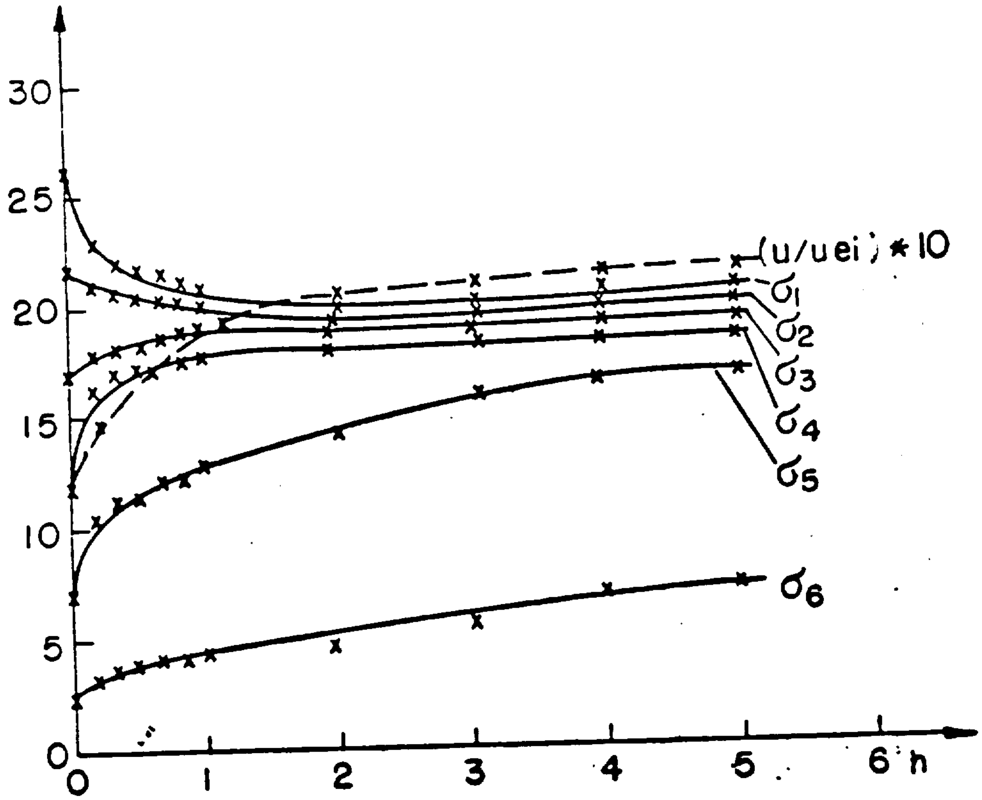

Fig. 6.4 - Stress Distribution

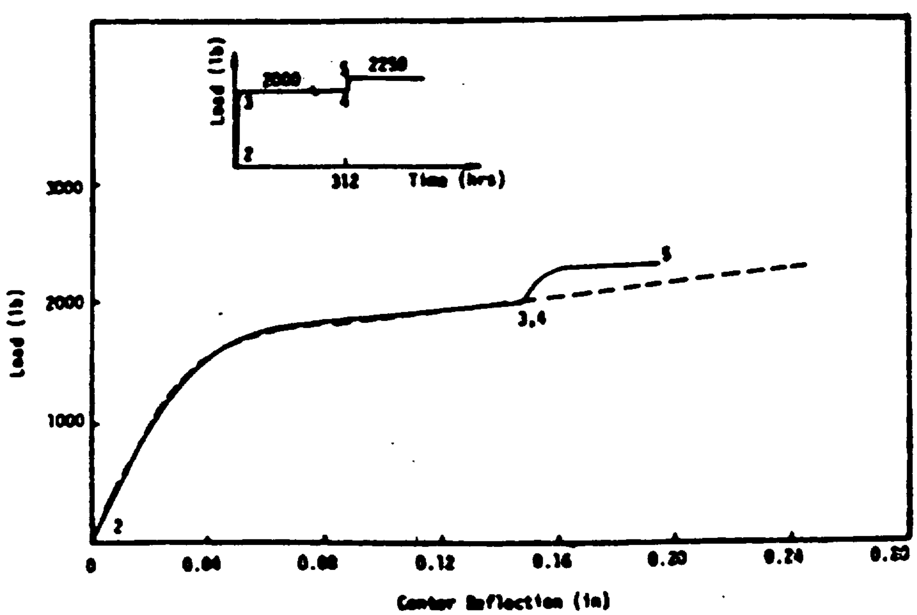

Fig. 6.5 - Deflection vs Load

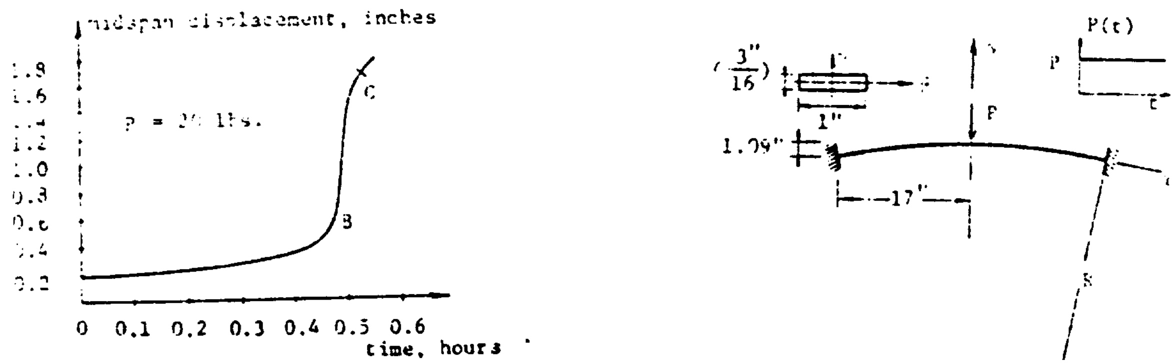

Fig. 6.6 - Circular Shallow Arch

7. FUTURE RESEARCH

As a consequence of these formulations, computational methods may be constructed. The incremental differential theorems lead to various finite difference methods. However, an integral theorem like the principle of the rate of virtual power calls for implementation by a finite element method. The discretization of the shell-like structure into finite elements and their systematic insertion into the integral theorems may yield a system of nodal motion differential equations. Numerous such applications are likely to be derived where large thermomechanical loads are anticipated.

To develop geometrically nonlinear, doubly curved finite shell elements the basic equations of nonlinear shell theories have to be transferred into the finite element model. As these equations in general are written in tensor notation their implementation into the finite element matrix formulation requires considerable effort. The next effort will concentrate how to derive the nonlinear element matrices directly from the incrementally formulated nonlinear shell equations using a tensor-oriented procedure. This enables the numerical realization of all structural responses, e.g. the calculation of pre- and post-buckling branches in snap-through analysis and especially in bifurcation analysis, including the detection of critical points and the consideration of geometric imperfections. To avoid loss of accuracy care will be taken for a realistic computation of the geometric properties as well as of the external loads. Finally, the developed family of shell elements will be presented and its efficiency will be demonstrated by some applications.

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13. ABSTRACT ***(Maximum 200 words)***

The object of the research reported herein was to develop a general mathematical model and solution methodologies for analyzing the structural response of thin, metallic shell structures under large transient, cyclic, or static thermo-mechanical loads. Among the system responses associated with these loads and conditions are thermal buckling, creep buckling, and ratcheting. Thus geometric and material nonlinearities (of high order) can be anticipated and must be considered in developing the mathematical model. The methodology is demonstrated through different problems of extension, shear and of planar curved beam. Moreover, importance of the inclusion of large strains is clearly demonstrated, through the chosen applications.

14. SUBJECT TERMS
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